ON THE EXISTENCE OF THE SOLUTION
FOR THE EQUATION
\[ f_1g_1 + f_2g_2 + \cdots + f_ng_n = 1 \]

MINORU MATSUBARA

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1. Introduction. Let \( X \) be a nonvoid set. We denote by \( BF(X) \) the set of all bounded functions from \( X \) to the complex field \( C \). If we define addition and multiplication to be the pointwise operations, \( BF(X) \) is a commutative algebra over \( C \). Let \( A \) be a subalgebra of \( BF(X) \) containing the constant function 1 which takes only the value 1 on \( X \). For \( f_1, f_2, \ldots, f_n \) in \( A \), let \( I(f_1, \ldots, f_n) \) be the ideal generated by \( f_1, \ldots, f_n \), i.e.,

\[ I(f_1, \ldots, f_n) = \{ f_1g_1 + \cdots + f_ng_n \mid g_1, \ldots, g_n \in A \}. \]

In this paper, we discuss the following problem: if \( f_1, \ldots, f_n \) in \( A \), what conditions must be imposed for \( f_1, \ldots, f_n \) to be \( I(f_1, \ldots, f_n) = A \)? Evidently \( I(f_1, \ldots, f_n) = A \) is equivalent to the following condition:

(1) There exist \( g_1, \ldots, g_n \) in \( A \) such that \( f_1g_1 + \cdots + f_ng_n = 1 \).

Then putting \( \delta^{-1} = \max\{|g_1|, \ldots, |g_n|\} \) where \( |g_i| = \sup\{|g_i(x)| \mid x \in X\} \), the above condition (1) implies the following condition:

(2) There exists a constant \( \delta > 0 \) such that \( |f_1(x)| + \cdots + |f_n(x)| \geq \delta \) for any \( x \) in \( X \).

As is easily seen, the condition (2) generally does not imply the condition (1). In the section 3 of this paper, we prove a necessary and sufficient condition on which the condition (2) implies the condition (1) when \( A \) is a Banach function algebra on \( X \).

These settings are a generalization of a theorem stated in [2] (Theorem 1) and a proposition stated in [3] (Corollary 2). Finally, in the section 4, we give a characterization of the polynomial convexity (Theorem 3) and this is an answer for one of problems which were raised by J. Wermer in [4].

2. Basic facts. Let \( A \) be a commutative Banach algebra over the complex field \( C \) with unit \( e \). We denote by \( \mathcal{M}(A) \) the set of all non trivial homomorphisms from \( A \) into \( C \). For each \( f \) in \( A \), we define the Gelfand transform \( \hat{f} \) of \( f \), \( \hat{f} : \mathcal{M}(A) \to C \), by \( \hat{f}(\phi) = \phi(f) \) for \( \phi \) in \( \mathcal{M}(A) \).

We denote by \( \hat{A} \) the set of all Gelfand transforms of \( A \). We define the Gelfand topology on \( \mathcal{M}(A) \) to be the weak \( \hat{A} \) topology, i.e., the weakest topology on \( \mathcal{M}(A) \) for which all the functions \( \hat{f}(f \in A) \) are continuous. Thus a neighborhood base at \( \phi_0 \) in \( \mathcal{M}(A) \) consists of sets
of the form

\[ U(\phi_0; g_1, \ldots, g_n, \varepsilon) = \{ \phi \in \mathcal{M}(A) \mid |\hat{\phi}_i(\phi) - \hat{\phi}_i(\phi_0)| < \varepsilon, \ i = 1, 2, \ldots, n \} \]

\[ (g_1, \ldots, g_n \in A \text{ and } \varepsilon > 0) \]

The following is well known:

**PROPOSITION 1.** (Gelfand Representation Theorem)
(i) \( \mathcal{M}(A) \) is a nonvoid compact Hausdorff space.
(ii) \( \hat{A} \) is the subalgebra of \( C(\mathcal{M}(A)) \) (= the algebra over \( C \) of all complex valued continuous functions defined on \( \mathcal{M}(A) \) under the pointwise operations) and \( \hat{A} \) separates the points in \( \mathcal{M}(A) \), i.e., for every \( \phi_1, \phi_2 \in \mathcal{M}(A), \phi_1 \neq \phi_2 \), there exists \( f \in A \) such that \( \hat{f}(\phi_1) \neq \hat{f}(\phi_2) \).
(iii) The Gelfand map \( \hat{\cdot} : A \to \hat{A} \) is a homomorphism and norm decreasing, i.e.,

\[ ||\hat{f}||_\infty = \sup \{ ||\hat{f}(\phi)||_1 \mid \phi \in \mathcal{M}(A) \} \leq ||f|| \]

(= the norm of \( f \) in \( A \)) for \( f \) in \( A \).

**PROPOSITION 2.** (Alternative Theorem)
For any \( f_1, \ldots, f_n \) in \( A \), the following alternatives hold:
(i) There exists a \( \phi \) in \( \mathcal{M}(A) \) such that \( \hat{f}_1(\phi) = \cdots = \hat{f}_n(\phi) = 0 \).
(ii) There exist \( g_1, \ldots, g_n \) in \( A \) such that \( f_1 g_1 + \cdots + f_n g_n = e \).

**Proof** ([1]). Let \( I = I(f_1, \ldots, f_n) = \{ f_1 g_1 + \cdots + f_n g_n \mid g_1, \ldots, g_n \in A \} \), then the following alternatives hold:
(a) The ideal \( I \) is proper.
(b) The ideal \( I \) is not proper, i.e., \( I = A \).

In case of (a), the assertion (i) readily holds (by using Zorn's Lemma).

(Q.E.D.)

**DEFINITION 1.** Let \( X \) be a nonvoid topological space. We say that \( A \subset C(X) \) is a Banach function algebra on \( X \) if:
(i) \( A \) is a commutative Banach algebra (with a norm \( || \cdot || \)) with unit 1.
(ii) \( A \) separates the points in \( X \).

Let \( A \) be a Banach function algebra on \( X \). Then \( X \) becomes necessarily a Hausdorff topological space. For each \( x \) in \( X \), if we define \( \tau(x) \) by \( \tau(x)f = f(x) \) for \( f \) in \( A \), then it is clear that \( \tau(x) \) is an element of \( \mathcal{M}(A) \) and the evaluation map \( \tau : X \to \mathcal{M}(A) \) is injective and continuous.

Each element \( f \) in \( A \) is bounded on \( X \), i.e., the following inequalities hold:
\[ ||f||_\infty \leq ||\hat{f}||_\infty \leq ||f|| \]
for \( f \) in \( A \),
because \( |f(x)| = |\hat{\tau}(x)| \leq ||\hat{f}||_\infty \leq ||f|| \) for \( x \) in \( X \) by Proposition 1.
DEFINITION 2. We say $A \subset C(X)$ is a uniform algebra on $X$ if:
(i) $X$ is a compact topological space.
(ii) $A$ is a Banach function algebra with respect to the $|| \cdot ||_\infty$ (the uniform norm).

If $A$ is a uniform algebra on $X$, then, because of $X$ being compact and $\mathcal{M}(A)$ being Hausdorff, the evaluation map $\tau: X \to \tau(X) \subset \mathcal{M}(A)$ is a homeomorphism.

3. THEOREM 1. Let $A$ be a Banach function algebra on $X$ and $\tau: X \to \mathcal{M}(A)$ be the evaluation map. Then the following conditions are equivalent:
(i) Suppose that $f_1, \cdots, f_m$ in $A$ satisfy $|f_i(x)| + \cdots + |f_m(x)| \geq \delta$ for some fixed $\delta > 0$ and for all $x$ in $X$. Then there exist $g_1, \cdots, g_m$ in $A$ such that $f_1g_1 + \cdots + f_mg_m = 1$.
(ii) $\tau(X)$ is dense in $\mathcal{M}(A)$.

Proof ([2]). (i) implies (ii). Suppose that there is a $\phi_0$ in $\mathcal{M}(A)$ which is not in the closure of $\tau(X)$. Then there is a neighborhood $U(\phi_0; g_1, \cdots, g_m, \epsilon)$ of $\phi_0$ which does not intersect $\tau(X)$. Therefore for any $x$ in $X$, there is an integer $k(1 \leq k \leq m)$ such that $|\hat{g}_k(\tau(x)) - \hat{g}_k(\phi_0)| \geq \epsilon$.

We put $f_i = g_i - \hat{g}_i(\phi_0)$ for $i=1, \cdots, m$. Then $f_i$ in $A$ for $i=1, \cdots, m$ and because of $\hat{f}_i(\tau(x)) = \hat{g}_i(\tau(x)) - \hat{g}_i(\phi_0)$, we have

$$|f_1(x)| + \cdots + |f_m(x)| \geq \epsilon$$

for all $x$ in $X$.

On the other hand, $\hat{f}_i(\phi_0) = 0$ for $i=1, \cdots, m$. Therefore, from Proposition 2, $I(f_1, \cdots, f_m) \neq A$.

(ii) implies (i). Suppose that $f_1, \cdots, f_m$ in $A$ satisfy $|f_1(x)| + \cdots + |f_m(x)| \geq \delta$ for some fixed $\delta > 0$ and for all $x$ in $X$.

Let $\phi \in \mathcal{M}(A)$ and $0 < \epsilon < \delta/n$. As $\tau(X)$ is dense in $\mathcal{M}(A)$, there is an $x$ in $X$ such that $\tau(x) \in U(\phi; f_1, \cdots, f_m, \epsilon)$, i.e.,

$$|\hat{f}_i(\tau(x)) - \hat{f}_i(\phi)| = |f_i(x) - \hat{f}_i(\phi)| < \epsilon$$

for $i=1, \cdots, n$.

Then $|f_i(x)| < |\hat{f}_i(\phi)| + \epsilon < |\hat{f}_i(\phi)| + \delta/n$ for $i=1, \cdots, n$, and

$$|\hat{f}_i(\phi)| + \cdots + |\hat{f}_n(\phi)| + \delta > |f_i(x)| + \cdots + |f_m(x)| \geq \delta.$$

Therefore, $|\hat{f}_i(\phi)| + \cdots + |\hat{f}_n(\phi)| > 0$.

Thus, again using Proposition 2, we can conclude $I(f_1, \cdots, f_m) = A$.

(Q.E.D.)

COROLLARY 2. Let $A$ be a Banach function algebra on a compact space $X$ and $\tau: X \to \mathcal{M}(A)$ be the evaluation map. Then the following conditions are equivalent:
(i) Suppose that $f_1, \cdots, f_m$ in $A$ do not have common zeros in $X$.
Then there exist $g_1, \cdots, g_m$ in $A$ such that $f_1g_1 + \cdots + f_mg_m = 1$.
(ii) $\tau(X) = \mathcal{M}(A)$.
Proof. It is clear from Theorem 1 by applying the fact that a continuous map from a compact space to a Hausdorff space preserves closed set. (Q.E.D.)

4. Let $C^n$ be the space of $m$-tuples of complex numbers, $X$ be a compact set in $C^n$ and $P(X)$ be the closure with respect to the uniform norm $\| \|_\infty$ in $C(X)$ of the polynomials in the coordinate functions. Then $P(X)$ is a uniform algebra on $X$.

Definition 3. Let $X$ be a compact set in $C^n$. We define the polynomially convex hull of $X$, denoted $\text{hull}(X)$, by

$$\text{hull}(X) = \{ z \in C^n | \| p(z) \|_\infty \leq \| p \|_\infty \text{ for every polynomial } p \}.$$  

$X$ is said to be polynomially convex if $\text{hull}(X) = X$.

Theorem 3. Let $X$ be a compact set in $C^n$. Then the following conditions are equivalent:

(i) Suppose that $f_1, \ldots, f_m$ in $P(X)$ do not have common zeros in $X$. Then there exist $g_1, \ldots, g_m$ in $P(X)$ such that $f_1g_1 + \cdots + f_ng_n = 1$.

(ii) $X$ is polynomially convex.

Before we shall prove Theorem 3, we need the following:

Lemma 4. A compact set $X$ in $C^n$ is polynomially convex, if and only if $\tau(X) = \mathfrak{M}(P(X))$.

Proof. Suppose that $X = \text{hull}(X)$. Let $\varphi$ be in $\mathfrak{M}(P(X))$, and put $\lambda = (\varphi(z_1), \ldots, \varphi(z_m))$ where $z_1, \ldots, z_m$ are the coordinate functions. For any non-negative integers $s_1, \ldots, s_m$, we have $(z_1^{s_1}z_2^{s_2}\cdots z_m^{s_m})(\lambda) = \varphi(z_1^{s_1}z_2^{s_2}\cdots z_m^{s_m})$. Hence we have $p(\lambda) = \varphi(p)$ for every polynomial $p$, and $|p(\lambda)| \leq \| \varphi \|_\infty \| p \|_\infty = \| p \|_\infty$. Therefore $\lambda$ is in $\text{hull}(X) = X$. By continuity, we have $\tau(\lambda)f = \varphi(f)$ for any $f$ in $P(X)$, by which we can conclude that $\tau(\lambda) = \varphi$. Conversely, we suppose that $\tau(X) = \mathfrak{M}(P(X))$. By Definition 3, it is clear that $X \subset \text{hull}(X)$ and for any, but fixed $\lambda$ in $\text{hull}(X)$, we have the following,

\[(\ast) \quad |p(\lambda)| \leq \| p \|_\infty \quad \text{for any polynomial } p.\]

So we define $\bar{\varphi}$ by $\bar{\varphi}(p) = p(\lambda)$ for any polynomial $p$. For $f$ in $P(X)$, there are polynomials $p_1, p_2, \ldots$ such that $p_n \to f$ uniformly on $X$. Then, using $(\ast)$, we have

$$|p_j(\lambda) - p_k(\lambda)| \leq \| p_j - p_k \|_\infty \to 0 \quad (j, k \to \infty).$$

Therefore $\lim_{n \to \infty} p_n(\lambda)$ exists, and again using $(\ast)$, we can prove that this limit is uniquely determined for $f$. Thus we can define $\varphi: P(X) \to C$, by $\varphi(f) = \lim_{n \to \infty} p_n(\lambda)$. It is clear that $\varphi$ is in $\mathfrak{M}(P(X))$.

Therefore, by our assumption, we can conclude that there exists a $\lambda_0$ in $X$ such that $\tau(\lambda_0) = \varphi$. For each coordinate function $z_j$, $j = 1, \ldots, m,$
we have \( \tau(\lambda_0)z_j = z_j(\lambda_0) = \varphi(z_j) = z_j(\lambda) \). Thus \( \lambda \) coincides with \( \lambda_0 \). Hence \( \lambda \) is in \( X \).

Proof of Theorem 3. Combine Corollary 2 with Lemma 4.

(Q.E.D.)

Bibliography