Doctoral Dissertation

DISTANCES BY LOCAL MOVES AND INVARIANTS OF VIRTUAL KNOTS

（局所変形による距離と仮想結び目の不変量）

November 25, 2013

MIGIWA SAKURAI

Graduate School of Science
Tokyo Woman's Christian University

Acknowledgements

I am most grateful to my thesis adviser Professor Yoshiyuki Ohyama at Tokyo Woman's Christian University for his constant encouragement and valuable advice. I express my heartfelt thanks to Professor Ryo Nikkuni at Tokyo Woman's Christian University and Professor Kouki Taniyama at Waseda University for their hearty encouragement.

Finally, I would like to show my hearty thanks to my family for supporting my school life.

Migiwa Sakurai
Division of Mathematics
Graduate School of Science
Tokyo Woman's Christian University
2-6-1, Zempukuji, Suginami-ku,
Tokyo, 167-8585,
Japan

Contents

Introduction

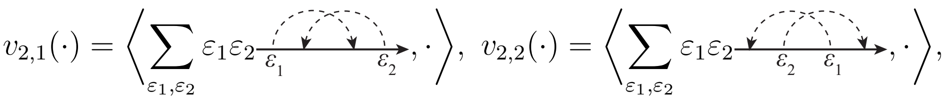
In 1999, Kauffman [8] introduced Virtual Knot Theory as a generalization of Knot Theory. Virtual knots are considered to have many interesting property different from classical knots which are usual knots in Knot Theory. In Knot Theory, there are many kinds of studies about the relation between knot invariants and local moves. We consider a local move on a knot diagram. The distance between two knots by the local move is defined to be the minimal number of times of the local move needed to transform one knot into the other knot. If the other knot is the trivial knot, the distance by the local move is called an unknotting number by the local move. In general, it is difficult to determine exact values of distances and unknotting numbers. The move called a crossing change is the most elementary unknotting operation in Knot Theory, and the unknotting number by crossing changes is a classical knot invariant (see [9]). It is well known that unknotting numbers for many classical knots by crossing changes can be determined from a knot signature (see [9]).

In Virtual Knot Theory, forbidden moves are an unknotting operation (see [7, 10]). In the same way as knots, we may define the distance between virtual knots by forbidden moves and the unknotting numbers of virtual knots by forbidden moves. To find properties on the distance and the unknotting number, we use the polynomial invariant $\mathbf{p}_t$, the smoothing invariant $\mathbf{S}$ and the gluing invariant $\mathbf{G}$ by Henrich [5]. Invariants $\mathbf{S}$ and $\mathbf{G}$ are represented by virtual knot diagrams, and they can induce concrete invariants. In particular, the polynomial invariant $\mathbf{p}_t$ is induced from $\mathbf{S}$ by using an invariant called an intersection index of two components flat virtual links. In this thesis, we give $\mathbf{S}(K)-\mathbf{S}(K')$ and $\mathbf{G}(K)-\mathbf{G}(K')$ for two virtual knots K and K' which can be transformed into each other by a single forbidden move. Then we can obtain the difference of the values obtained from invariants induced from $\mathbf{S}$ and $\mathbf{G}$ between K and K'. In particular, we have

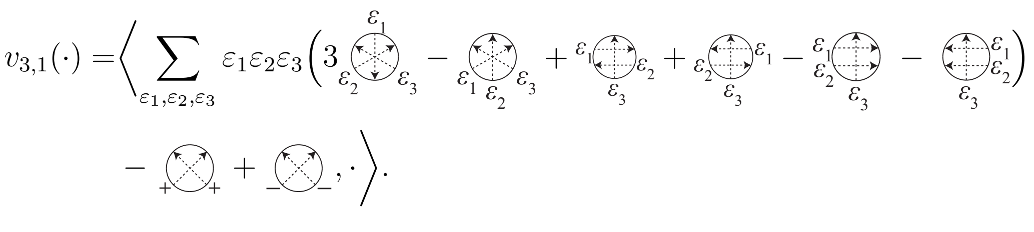
$$\mathbf{p}_t(K) - \mathbf{p}_t(K') = (t-1)(\pm t^k \pm t^l),$$

where k and ℓ are some integers. By the result for $\mathbf{p}_t(K)$, we can estimate the distance between two virtual knots by forbidden moves, and the unknotting number of a virtual knot by forbidden moves. Actually, we determine unknotting numbers of 54 virtual knots out of 117 virtual knots with up to four real crossing points.

In Knot Theory, Vassiliev [13] defined a finite type invariant. A finite type invariant is closely related to a local move called a C_n-move. We can calculate the difference of the values of the finite type invariant of degree n between two knots which can be transformed into each other by a C_n-move (see [11] and [12]). In Virtual Knot Theory, Goussarov, Polyak and Viro [3] defined a finite type invariant and a local move called an n-variation. They showed the following two formulas generate the finite type invariants of degree 2 for long virtual knots:



and the following formula generates the finite type invariants of degree 3 for virtual knots where $\varepsilon_i = \pm 1$ $(i = 1, 2, 3)$:



We think that finite type invariants and n-variations have relationships in Virtual Knot Theory. In this thesis, we give the differences of the values of $v_{2,1}$ and $v_{2,2}$ between two long virtual knots which can be transformed into each other by a 2-variation, and the difference of the values of $v_{3,1}$

between two virtual knots which can be transformed into each other by a 3-variation(3). By the results, we can obtain an estimate of the distance between long virtual knots by 2-variations, and the distance between virtual knots by 3-variation(3)'s. In addition, we consider the relation between an n-variation and the polynomial invariant $\mathbf{p}_t$ defined by Henrich [5]. Since a forbidden move is a 2-variation (see [3]), we obtain the difference of the values of $\mathbf{p}_t$ between two virtual knots which can be transformed into each other by a 2-variation as mentioned above. Moreover, we give the difference of the values of $\mathbf{p}_t$ between two virtual knots which can be transformed into each other by an n-variation(n) ($n \geq 3$).

This thesis is organized as follows. In Chapter 1, we review some basic notions of Virtual Knot Theory. In Chapter 2, we obtain the difference of the values of $\mathbf{p}_t$ between two virtual knots which can be transformed into each other by a forbidden move. Then, in Chapter 3, we give the differences of the values of $v_{2,1}$ and $v_{2,2}$ between two long virtual knots which can be transformed into each other by a 2-variation, and the difference of the values of $v_{3,1}$ between two virtual knots which can be transformed into each other by a 3-variation(3).

Chapter 1

Preliminaries

In this chapter, we introduce some definitions and notations.

1.1 Diagrams and equivalence classes

We introduce some diagrams on $\mathbb{S}^2$ and their equivalence classes. Here, all diagrams are oriented. A virtual knot diagram is presented by a knot diagram having virtual crossings as well as real crossings in Fig. 1.1.1. Two virtual knot diagrams are equivalent if one can be obtained from the other by a finite sequence of generalized Reidemeister moves in Fig. 1.1.3. The equivalence class of virtual knot diagrams modulo the generalized Reidemeister moves is called a *virtual knot*. A virtual string link diagram with μ strings and a long virtual knot diagram have virtual crossings in the same way as a virtual knot diagram. A *virtual string link* with μ strings and a *long virtual knot* are defined as a virtual knot. In the other words, they are the equivalence classes of their diagrams under the generalized Reidemeister moves. A flat virtual link diagram is a virtual link diagram without over and under information for each real crossing. A *flat virtual link* is an equivalence class of flat virtual link diagrams modulo the generalized Reidemeister moves without over and under information. A flat singular virtual link diagram is a flat virtual link diagram with singular crossings in Fig. 1.1.1. A *flat singular virtual link* is an equivalence class of flat singular virtual link diagrams modulo flat versions of the generalized Reidemeister moves and the flat singularity moves shown in Fig. 1.1.2.

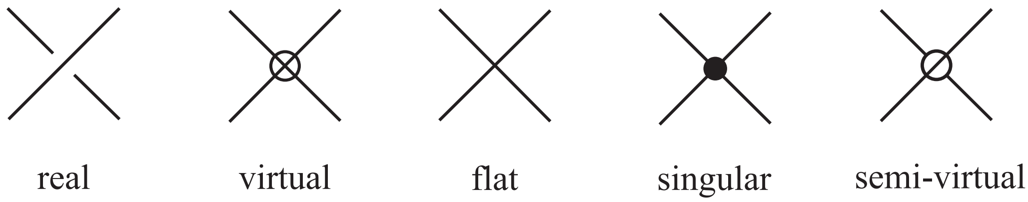


Figure 1.1.1 Crossing types

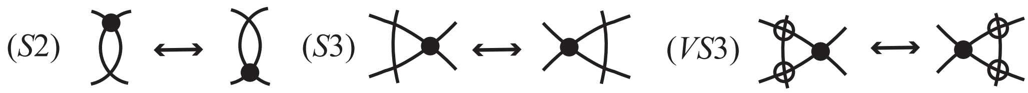


Figure 1.1.2 Flat singularity moves

Virtual knots, virtual string links with μ strings, and long virtual knots can be encoded by their Gauss diagrams. A virtual knot diagram, a virtual string link diagram with μ strings, and a long virtual knot diagram can

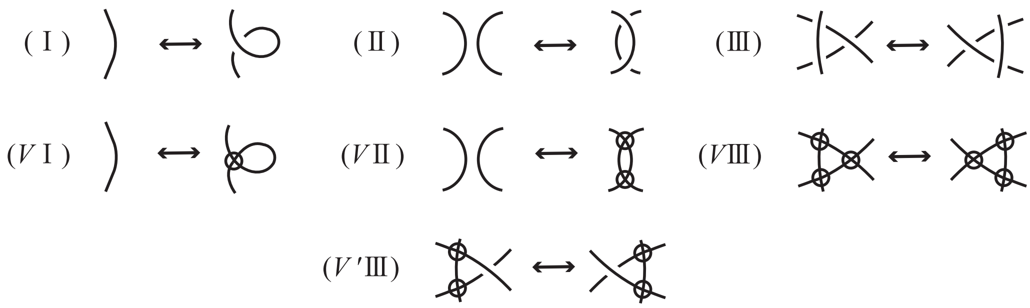


Figure 1.1.3 Generalized Reidemeister moves

be regarded as the image of an immersion from $\mathbb{S}^1$ into $\mathbb{R}^2$, that from unit intervals I_k $(k = 1, 2, \ldots, \mu)$ into $\mathbb{R}^2$, and that from $\mathbb{R}$ into $\mathbb{R}^2$, respectively. Let D be one of these diagrams. The *Gauss diagram* for D is the preimage of D with chords connecting the preimages of each real crossing. We specify the real crossing information on each chord by directing the chord toward the under crossing and decorating each chord with the sign of the crossing (Fig. 1.1.4).

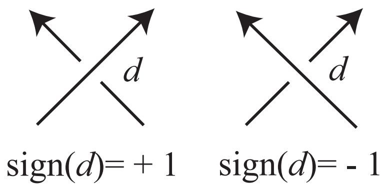


Figure 1.1.4 The sign of a real crossing

It is well known that there exists a bijection from all virtual knots to all equivalence classes of their Gauss diagrams under the generalized Reidemeister moves in Fig. 1.1.5. Then we can identify a virtual knot with it's Gauss diagram. The same results hold for a virtual string link, and a long virtual knot.

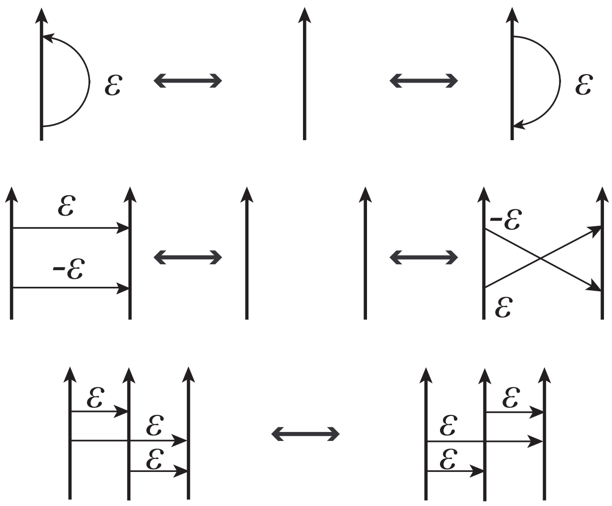


Figure 1.1.5 Generalized Reidemeister moves of Gauss diagrams

1.2 Local moves

Both of the moves on a virtual knot diagram depicted in Fig. 1.2.1 are called *forbidden moves*, and denoted by F. Forbidden moves are presented by local moves of Gauss diagrams in Fig. 1.2.2.

Figure 1.2.1 Forbidden moves

Kanenobu and Nelson showed Theorem 1.2.1 independently.

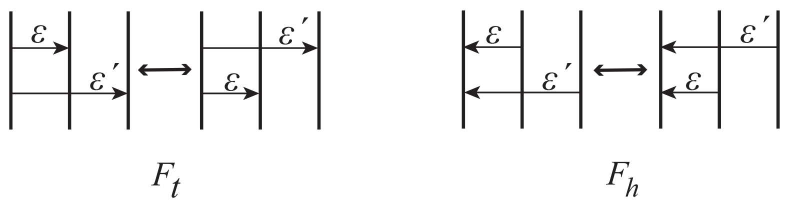


Figure 1.2.2 Forbidden moves of Gauss diagrams

Theorem 1.2.1 ([7], [10])**.** *Any virtual knot diagram can be deformed into any other virtual knot diagram by using forbidden moves and generalized Reidemeister moves.*

By Theorem 1.2.1, we can define the distance between any two virtual knots by using forbidden moves.

Definition 1.2.2. Let K and K' be virtual knots, and D and D' virtual knot diagrams of K and K' respectively. If a virtual knot diagram D can be transformed into D' by a set of generalized Reidemeister moves and local moves denoted by M, we denote the minimal number of times of M needed to transform D into D' by $d_M(K, K')$ and call it the *distance* between K and K' by M. In particular, if D' is the trivial knot diagram, it is denoted by $u_M(K)$ and called the *unknotting number* of K by M.

We note that the unknotting number $u_M(K)$ is a virtual knot invariant of K.

Chapter 2

Unknotting numbers by forbidden moves

2.1 Forbidden moves and Henrich's invariants

We recall the definition of invariants $\mathbf{S}(K)$, $\mathbf{G}(K)$ and $\mathbf{p}_t(K)$ for a virtual knot K by Henrich [5].

Definition 2.1.1 ([5])**.** Let D be a diagram of K, and $C(D)$ the set of all real crossings of D. Denote by D_g^d the flat singular virtual link diagram which is obtained from D by changing a real crossing $d \in C(D)$ for a singular crossing and ignoring over and under information for the other real crossings. Let $[D_g^d]$ be the flat singular equivalence class of D_g^d, and $[D_g^0]$ the flat singular equivalence class of the flat singular virtual knot with one singular crossing obtained by applying a generalized Reidemeister move (I) to D and exchanging the crossing for a singular crossing. The gluing invariant $\mathbf{G}(K)$ is given by

$$\mathbf{G}(K) = \sum_{d \in C(D)} \operatorname{sign}(d)\Big([D_g^d] - [D_g^0]\Big),$$

where sign(d) is the sign of d as in Fig. 1.1.4.

Furthermore, denote by D_s^d the flat virtual link diagram which is obtained from a virtual link diagram smoothed at d by ignoring over and under information for the other real crossings. Let $[D_s^d]$ be the flat equivalence class of D_s^d, and $[D_s^0]$ the flat equivalence class of the disjoint union of a flat diagram of D and a trivial knot. The smoothing invariant $\mathbf{S}(K)$ is given by

$$\mathbf{S}(K) = \sum_{d \in C(D)} \operatorname{sign}(d)\Big([D_s^d] - [D_s^0]\Big).$$

Invariants $\mathbf{p}_t(K)$ is defined in analogy with $\mathbf{S}(K)$. Let $D_s^d = D_1 \cup D_2$, and $1 \cap 2$ the set of all flat crossings between D_1 and D_2. For a flat crossing $e \in 1 \cap 2$, sgn(e) is the sign of e as in Fig. 2.1.1. The intersection index of D_s^d, $i(D_s^d)$, is defined by

$$i(D_s^d) = \sum_{e \in 1 \cap 2} \operatorname{sgn}(e).$$

Since the value of $i(D_s^d)$ is depend on d, $i(D_s^d)$ can be also denoted by $i(d)$. Then the polynomial invariant $\mathbf{p}_t(K)$ is given by

$$\mathbf{p}_t(K) = \sum_{d \in C(D)} \operatorname{sign}(d)\left(t^{|i(d)|} - 1\right).$$

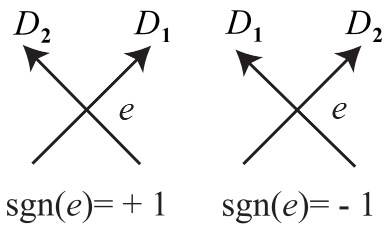


Figure 2.1.1 The sign of a flat crossing

Remark 2.1.2. Let $\mathbb{Z}\mathcal{L}$ denote the free abelian group generated by a set $\mathcal{L}$ of all 2-component flat virtual links. Define a map $\phi\colon \mathcal{L} \to \mathbb{Z}[t]$ to be the map such that $\phi(L) = t^{|i(L)|}$ for any 2-component flat virtual link L. Extend it to $\mathbb{Z}\mathcal{L}$ linearly. Then $\mathbf{p}_t = \phi \circ \mathbf{S}$. In this way, $\mathbf{S}(K)$ and $\mathbf{G}(K)$ induce invariants for K by using invariants of 2-component flat virtual links and 2-component flat singular virtual links respectively.

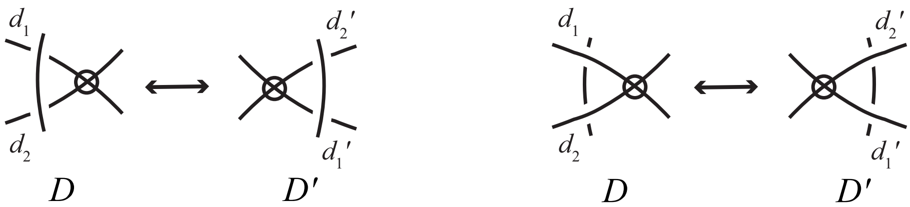


Figure 2.1.2

From here, we consider forbidden moves and invariants for virtual knots. Let K and K' be virtual knots represented by diagrams D and D' respectively as shown in Fig. 2.1.2. The diagram D is obtained from D' by a single forbidden move. Let d_i ($i = 1, 2, \ldots, n$) be the real crossings of K and

d_i' $(i = 1, 2, \ldots, n)$ the real crossings of K' corresponding to d_i. By the definition,

$$\mathbf{S}(K) - \mathbf{S}(K') = \sum_{j=1}^{n} \mathrm{sign}(d_j) \left\{ ([D_s^{d_j}] - [D_s'^{d_j'}]) - ([D_s^0] - [D_s'^0]) \right\}, \quad (2.1.1)$$

$$\mathbf{G}(K) - \mathbf{G}(K') = \sum_{j=1}^{n} \mathrm{sign}(d_j) \left\{ ([D_g^{d_j}] - [D_g'^{d_j'}]) - ([D_g^0] - [D_g'^0]) \right\}. \quad (2.1.2)$$

Theorem 2.1.3. *For* $\mathbf{p_t}(K)$, *we have*

$$\mathbf{p_t}(K) - \mathbf{p_t}(K') = \begin{cases} (t-1)\left(\pm t^{|i(d_1)|} \pm t^{|i(d_2)|}\right) \\ (t-1)\left(\pm t^{|i(d_1)|} \pm t^{|i(d_2)|-1}\right) \\ (t-1)\left(\pm t^{|i(d_1)|-1} \pm t^{|i(d_2)|}\right) \\ (t-1)\left(\pm t^{|i(d_1)|-1} \pm t^{|i(d_2)|-1}\right) \end{cases}. \quad (2.1.3)$$

Proof. Due to (2.1.1) and Remark 2.1.2,

$$\mathbf{p_t}(K) - \mathbf{p_t}(K') = \sum_{j=1}^{n} \mathrm{sign}(d_j) \left(t^{|i(d_j)|} - t^{|i(d_j')|} \right).$$

We consider the terms of $\mathbf{p}_t(K) - \mathbf{p}_t(K')$ corresponding to d_k and d_k' $(3 \leq k \leq n)$. Denote by $\widetilde{d}$ the flat crossing corresponding to a real crossing d. $D_s^{d_k}$ and $D_s'^{d_k'}$ are identical except for d_1, d_2, d_1' and d_2'. Since $\mathrm{sgn}(\widetilde{d_1}) = \mathrm{sgn}(\widetilde{d_1'})$ and $\mathrm{sgn}(\widetilde{d_2}) = \mathrm{sgn}(\widetilde{d_2'})$, $|i(d_k)| = |i(d_k')|$. Therefore, $\mathrm{sign}(d_k)(t^{|i(d_k)|} - t^{|i(d_k')|}) = 0$.

Now, we consider the terms of $\mathbf{p}_t(K) - \mathbf{p}_t(K')$ corresponding to d_ℓ and d_ℓ' $(\ell = 1, 2)$. Figure 2.1.3 illustrates all cases of $D_s^{d_\ell}$ and $D_s'^{d_\ell'}$. If the string 3 belongs to the same component as the string 1, $\widetilde{d_m}$ does not contribute $|i(d_\ell)|$

and $\widetilde{d'_m}$ contributes $|i(d'_\ell)|$ ($\ell \neq m$ and $m = 1,\, 2$). On the other hand, if the string 3 belongs to the same component as the string 2, $\widetilde{d_m}$ contributes $|i(d_\ell)|$ and $\widetilde{d'_m}$ does not contribute $|i(d'_\ell)|$. Thus $|i(d_\ell)| = |i(d'_\ell)| \pm 1$. Therefore,

$$\begin{aligned}
&\mathrm{sign}(d_1)\left(t^{|i(d_1)|} - t^{|i(d'_1)|}\right) + \mathrm{sign}(d_2)\left(t^{|i(d_2)|} - t^{|i(d'_2)|}\right) \\
&= \pm t^{|i(d_1)|}\left(1 - t^{\pm 1}\right) \pm t^{|i(d_2)|}\left(1 - t^{\pm 1}\right) \\
&= \begin{cases} (t-1)\left(\pm t^{|i(d_1)|} \pm t^{|i(d_2)|}\right) \\ (t-1)\left(\pm t^{|i(d_1)|} \pm t^{|i(d_2)|-1}\right) \\ (t-1)\left(\pm t^{|i(d_1)|-1} \pm t^{|i(d_2)|}\right) \\ (t-1)\left(\pm t^{|i(d_1)|-1} \pm t^{|i(d_2)|-1}\right) \end{cases}.
\end{aligned}$$

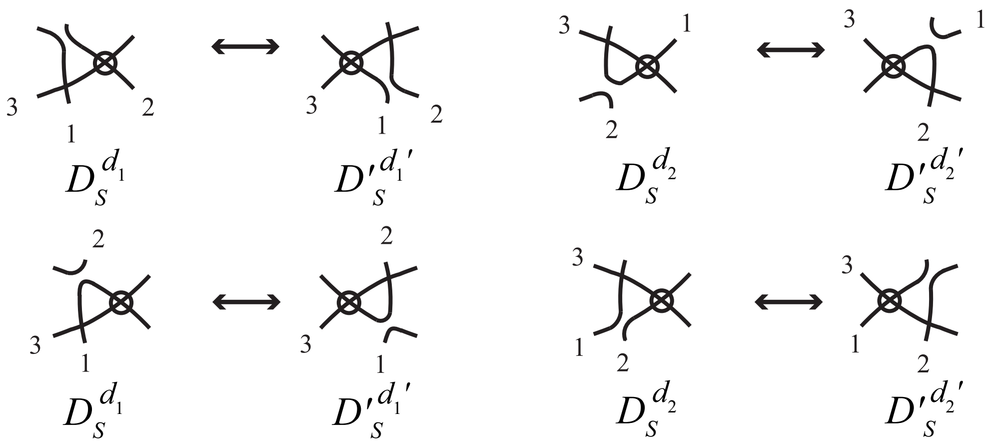


Figure 2.1.3

□

Corollary 2.1.4. *Let K and K' be virtual knots, and $\mathbf{p}_t(K) - \mathbf{p}_t(K') = (t-1)\sum_{j\geq 0} a_j t^j$. Then,*

$$d_F(K, K') \geq \frac{\sum_{j\geq 0} |a_j|}{2}.$$

In particular, let $p_t(K) = (t-1)\sum_{k\geq 0} b_k t^k$ *for a virtual knot* K*. Then we have*

$$u_F(K) \geq \frac{\sum_{k\geq 0} |b_k|}{2}.$$

2.2 Examples

Let D be a virtual knot diagram. An n-bridge presentation of D is a division of D into n overbridges (paths without under crossings) and n underbridges (paths without over crossings) appearing alternately along the diagram. The bridge number $b(K)$ of a virtual knot K is the minimal number of overbridges of all bridges presentations of the diagrams representing K ([1], [6]).

Example 2.2.1. For $a, b \in \mathbb{N}$, let K be virtual knot represented by the diagram D as shown in Fig. 2.2.1. Since $D_s^{c_j}$ $(1 \leq j \leq a)$ and $D_s^{c_k}$ $(a+1 \leq k \leq a+b)$ are flat virtual links as shown in Fig. 2.2.2, $|i(c_j)| = b$ and $|i(c_k)| = a$. Then we have

$$\begin{aligned}\mathbf{p}_t(K) =& a(t^b - 1) + b(t^a - 1)\\ =&(t-1)\{a(t^{b-1} + t^{b-2} + \cdots + 1) + b(t^{a-1} + t^{a-2} + \cdots + 1)\}.\end{aligned}$$

From Corollary 2.1.4, $u_F(K) \geq ab$.

The virtual knot K is presented by the Gauss diagram G in Fig. 2.2.3. We perform forbidden moves on the leftmost vertical chord in G b times, and obtain the diagram G' as in Fig. 2.2.3. The diagram G' has $a-1$ vertical chords and b horizontal chords. By repeated use of this move a times, G may be changed to a Gauss diagram with b horizontal chords. These chords are removed via generalized Reidemeister moves (I) for Gauss diagrams. Therefore $u_F(K) = ab$.

From the above arguments, we see that there is a virtual knot K such that $u_F(K) = n$ and $b(K) = 1$ for any $n \in \mathbb{N}$.

Example 2.2.2. Table 3.3.1 shows all virtual knots with up to 4 real crossing points. We can determine unknotting numbers of 54 virtual knots as in Tab. 2.2.1.

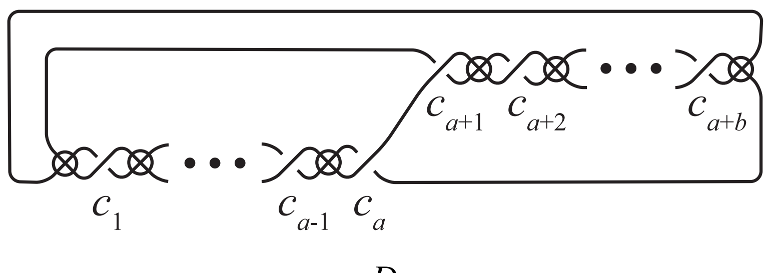


Figure 2.2.1

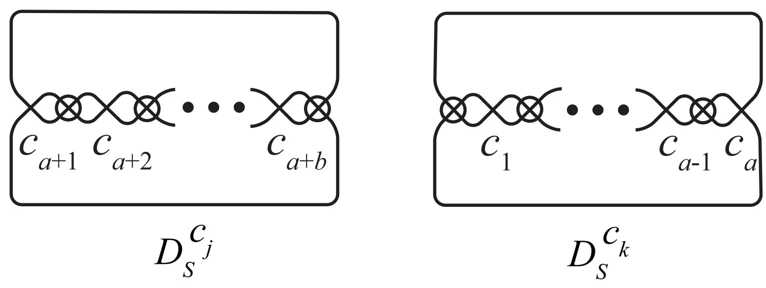


Figure 2.2.2

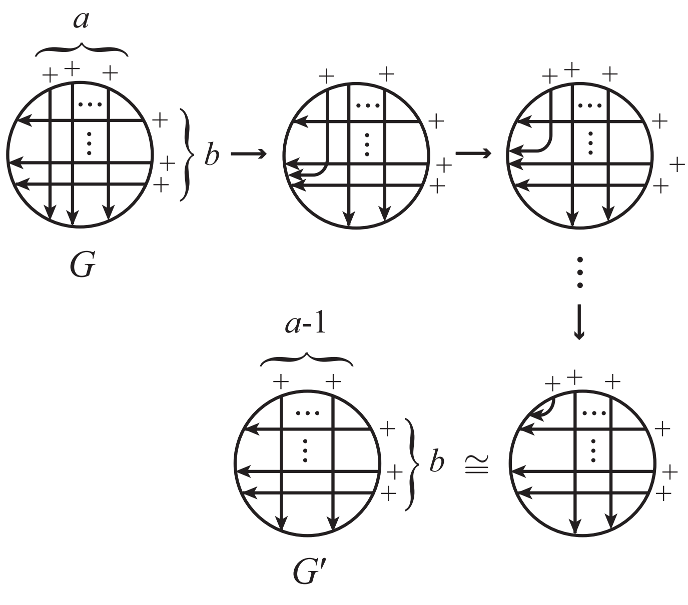


Figure 2.2.3

K	$u_F(K)$	K	$u_F(K)$
0.1	0	4.37	3
2.1	1	4.38	1
3.1	1	4.39	1
3.2	1	4.40	1
3.3	2	4.42	1
3.4	1	4.43	2
4.1	2	4.45	2
4.3	2	4.48	3
4.4	1	4.49	1
4.5	1	4.50	1
4.7	2	4.52	1
4.11	2	4.53	2
4.15	2	4.54	1
4.17	1	4.57	1
4.18	1	4.60	1
4.20	1	4.63	2
4.21	2	4.64	1
4.22	1	4.73	2
4.23	1	4.74	1
4.25	2	4.79	1
4.28	2	4.80	3
4.29	2	4.81	2
4.32	1	4.82	3
4.33	1	4.83	2
4.34	1	4.88	1
4.35	1	4.89	4
4.36	2	4.93	2

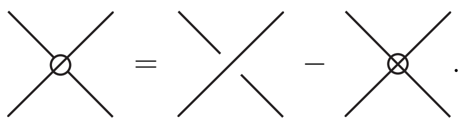
Table 2.2.1 Unknotting numbers

Chapter 3

2- and 3-variations and finite type invariants of degree 2 and 3

3.1 n-variations and finite type invariants

We recall the finite type invariant based on the study by Goussarov, Polyak and Viro. Virtual knot diagrams (or long virtual knot diagrams) are extended to diagrams with semi-virtual crossings in Fig. 1.1.1. Semi-virtual crossings are related to the other crossings by the following relation in a free abelian group $\mathbb{Z}[\mathcal{K}]$ generated by the set $\mathcal{K}$ of all virtual knots (or long virtual knots):



Let D be a virtual knot diagram (or a long virtual knot diagram), and $(d_1, d_2, \ldots, d_n)$ an n-tuple of real crossings of D. For an n-tuple $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ of 0 and 1, define D_δ to be the diagram obtained from D by switching the crossing d_i with $\delta_i = 1$ to a virtual crossing. Denote by $|\delta|$ the number of 1 in δ. The following sum is called a diagram with n semi-virtual crossings and denoted by D^n:

$$\sum_{\delta}(-1)^{|\delta|}D_{\delta}.$$

Definition 3.1.1 ([3])**.** Put $v : \mathcal{K} \to G$ to be an invariant of virtual knots with values in an abelian group G. Extend it to $\mathbb{Z}[\mathcal{K}]$ linearly. Then the invariant v is called a *finite type invariant* of degree n, if the equality $v(K^m) = 0$ holds for all diagrams K^m with m semi-virtual crossings $(m > n)$.

An arrow diagram is just a Gauss diagram with all chords drawn dashed. Let $\mathcal{A}$ be the set of all arrow diagrams, and $\mathcal{G}$ the set of all Gauss diagrams. A subdiagram of $G \in \mathcal{G}$ is a Gauss diagram consisting of some subset of the chords of G. Define a map $i : \mathcal{G} \to \mathcal{A}$ by the map which makes all the chords of a Gauss diagram dashed, and $I : \mathcal{G} \to \mathbb{Z}\mathcal{A}$ by

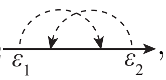
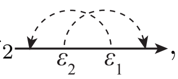
$$I\left(G\right) = \sum_{G' \subset G} i(G'),$$

where the sum is over all subdiagrams of G. Extend these to $\mathbb{Z}\mathcal{G}$ linearly. On the generators of $\mathbb{Z}\mathcal{A}$, define (G, H) to be 1 if $G = H$ and 0 otherwise, and then extend $(\cdot, \cdot)$ bilinearly. Put

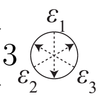
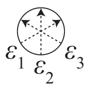
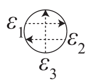
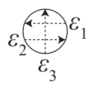
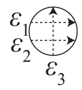
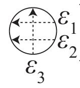
$$\langle A, G\rangle = \left(A, I\left(G\right)\right),$$

for any $G \in \mathcal{G}$ and $A \in \mathbb{Z}\mathcal{A}$. Then, the following results hold for the finite type invariant of low degree.

Proposition 3.1.2 ([3])**.** *Denote by v_n a $\mathbb{Z}$-valued finite type invariant of degree n. The invariant v_1 is a constant map. If $\mathcal{K}$ is the set of all virtual knots, then there is not v_2. On the other hand, if $\mathcal{K}$ is the set of all long virtual knots, then v_2 is generated by*

$v_{2,1}(\cdot) = \left\langle \sum_{\varepsilon_1,\varepsilon_2} \varepsilon_1\varepsilon_2$  $, \cdot \right\rangle$ *and* $v_{2,2}(\cdot) = \left\langle \sum_{\varepsilon_1,\varepsilon_2} \varepsilon_1\varepsilon_2$  $, \cdot \right\rangle.$

And, if $\mathcal{K}$ is the set of all virtual knots, then v_3 is generated by

$v_{3,1}(\cdot) = \left\langle \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3} \varepsilon_1\varepsilon_2\varepsilon_3 \Big(3$  $-$  $+$  $+$  $-$  $-$  $\Big)$

$-$  $+$  $, \cdot \right\rangle,$

where $\varepsilon_i = \pm 1$ $(i = 1, 2, 3)$.

An n-variation is defined by the similar way for a C_n-move.

Definition 3.1.3 ([3])**.** Let G be a Gauss diagram of a virtual string link with μ strings, and A_1, A_2, $\ldots$, A_{n+1} be the non-empty sets of chords of G. Then the Gauss diagram G is called *n-trivial*, if the following conditions are satisfied:

(i) $A_i \cap A_j = \emptyset$ $(i \neq j)$ and

(ii) we can change G into the Gauss diagram with no chords by applying generalized Reidemeister moves (II) of Gauss diagrams, if we remove from G all chords which belong to any non-empty subfamily of $\{A_1, A_2, \ldots, A_{n+1}\}$.

Let G' be a Gauss diagram of a virtual knot. Choose μ segments which do not contain an endpoint of any chord on the circle of G', and attach μ strings of G on these segments. This move is called an $n+1$-*variation*.

Forbidden moves are 2-variations (see [3]). From the above definition, an example of an n-variation is given as follows.

Example 3.1.4. The move depicted in Fig. 3.1.1 is an n-variation. It is called an *n-variation*(n) and denoted by (n).

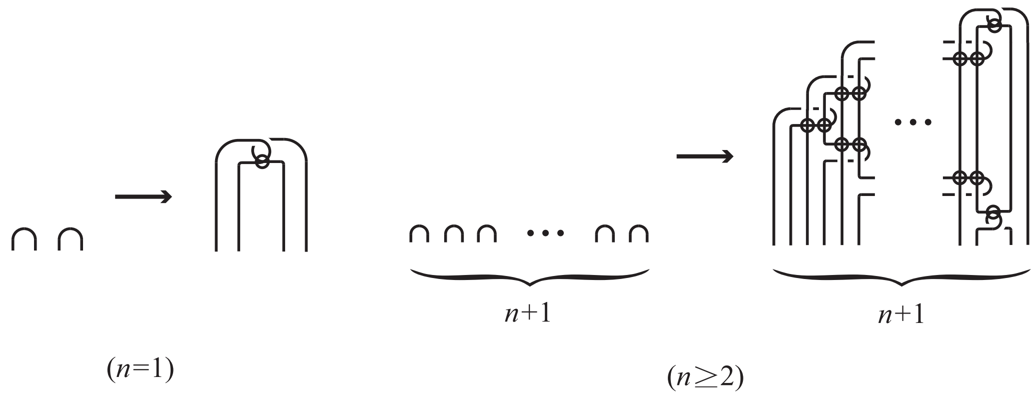


Figure 3.1.1 An n-variation(n)

Two local moves M_1 and M_2 are *equivalent*, if M_i can be realized by a single M_j ($i, j = 1, 2$ and $i \neq j$). The moves in Fig. 3.1.1 are equivalent to those in Fig. 3.1.2.

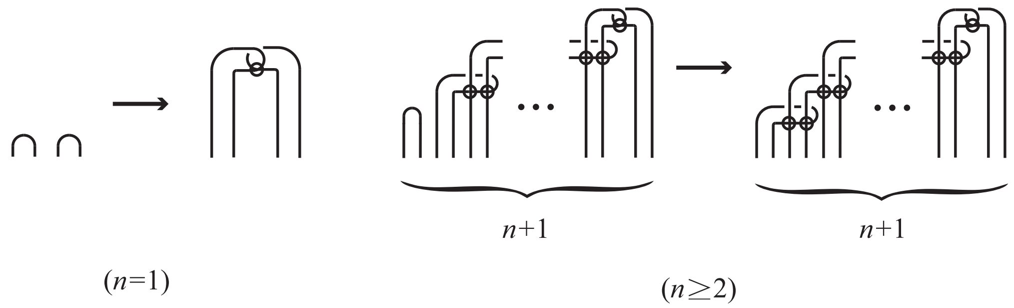


Figure 3.1.2 Moves equivalent to the moves in Fig. 3.1.1

3.2 2- and 3-variations and finite type invariants of degree 2 and 3

As described in Chapter1, Section1.2, forbidden moves are presented by local moves of Gauss diagrams in Fig. 1.2.2.

Lemma 3.2.1. *Any oriented forbidden move is realized by the moves in Fig. 3.2.1.*

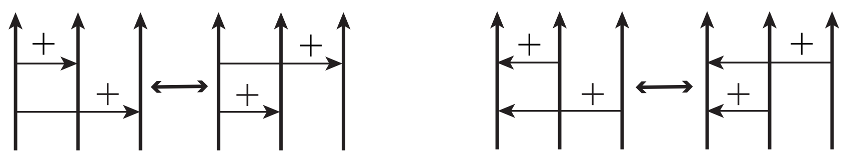


Figure 3.2.1

Proof. We give orientations to strings of F_t as in Fig. 3.2.2. The case of $\varepsilon, \varepsilon' = +1$ realizes the other cases as shown in Fig. 3.2.3.

Similarly, we can show that any oriented F_h is realized by the right move in Fig. 3.2.1.

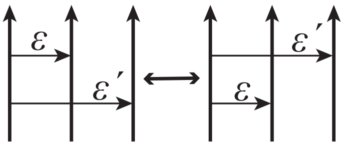


Figure 3.2.2

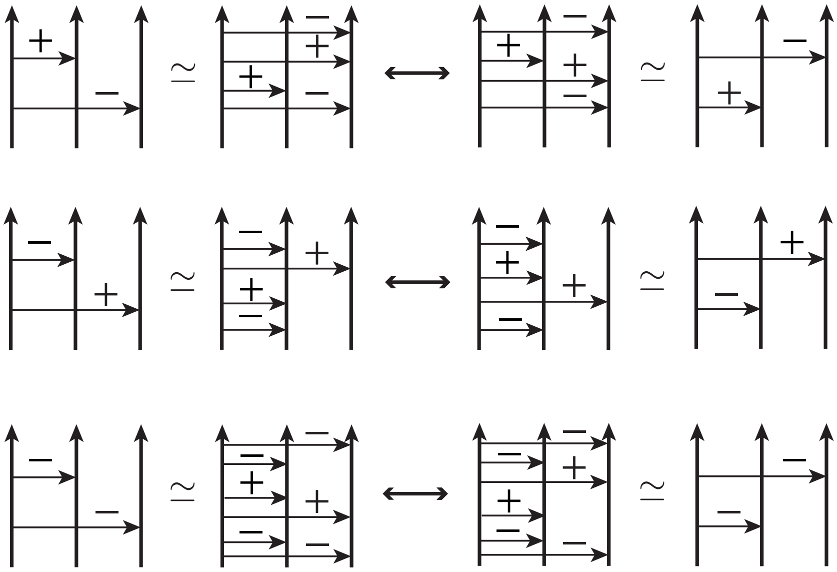


Figure 3.2.3

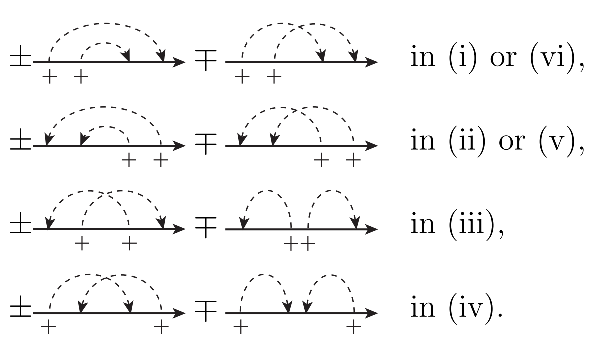
□

Theorem 3.2.2. *Let G and G' be Gauss diagrams of long virtual knots K*

and K' which can be transformed into each other by a single forbidden move respectively. Then we have

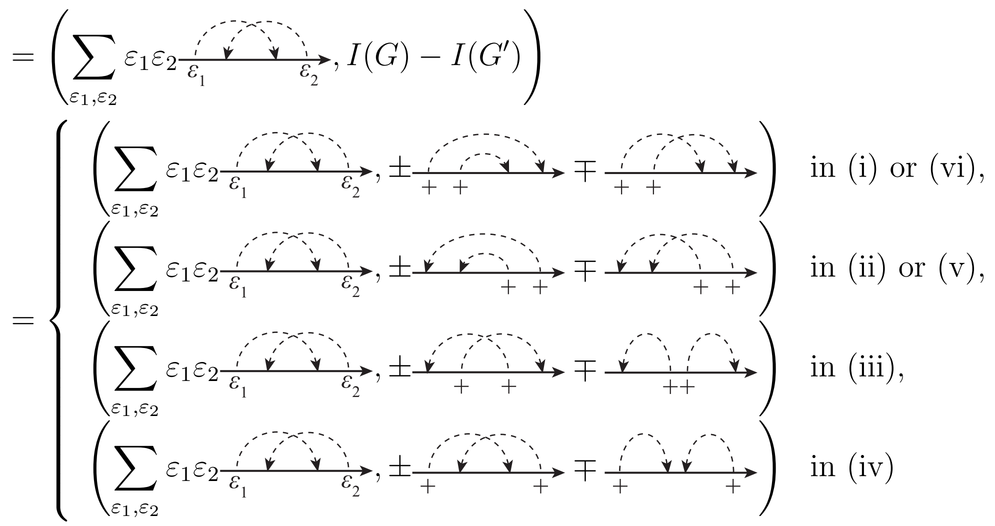
$$\big(v_{2,1}(K) - v_{2,1}(K'),\ v_{2,2}(K) - v_{2,2}(K')\big) = (0,\ 0),\ (0,\ \pm 1)\ \textit{or}\ (\pm 1,\ 0).$$

Proof. We consider the moves in Fig. 3.2.1 for Gauss diagrams of long virtual knots. It is enough to check the moves in Fig. 3.2.4. Let G and G' be Gauss diagrams which can be transformed into each other by a single move in Fig. 3.2.4. Let ℓ_1 and ℓ_2 be the two chords in the part where a forbidden move is applied. Since the subdiagrams of G with either ℓ_1 or ℓ_2 and those with neither ℓ_1 nor ℓ_2 are equal to the subdiagrams of G' with either ℓ_1 or ℓ_2 and those with neither ℓ_1 nor ℓ_2 respectively, these terms cancel each other in $I(G) - I(G')$. The following are the terms in $I(G) - I(G')$ corresponding to subdiagrams which consist of two chords and have both ℓ_1 and ℓ_2:



Therefore,

$$v_{2,1}(K) - v_{2,1}(K')$$



$$= \begin{cases} 0 & \text{in (i) or (vi),} \\ 0 & \text{in (ii) or (v),} \\ 0 & \text{in (iii),} \\ \pm 1 & \text{in (iv).} \end{cases}$$

Similarly,

$$v_{2,2}(K) - v_{2,2}(K') = \begin{cases} 0 & \text{in (i) or (vi),} \\ 0 & \text{in (ii) or (v),} \\ \pm 1 & \text{in (iii),} \\ 0 & \text{in (iv).} \end{cases}$$

□

Corollary 3.2.3. *Let K and K' be long virtual knots. Then,*

$$d_F(K, K') \geq \left|v_{2,1}(K) - v_{2,1}(K')\right| + \left|v_{2,2}(K) - v_{2,2}(K')\right|.$$

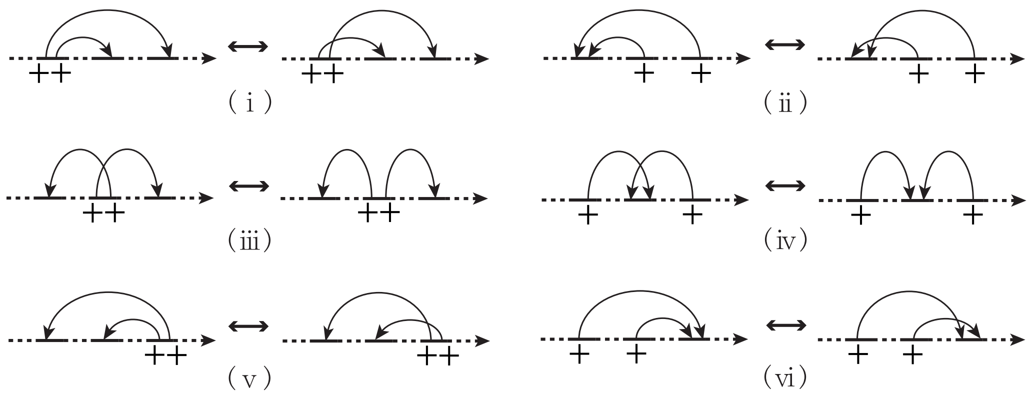
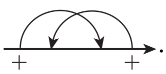
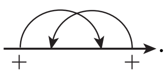


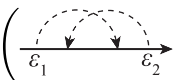
Figure 3.2.4 Gauss diagrams G and G'

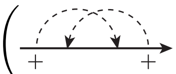
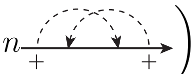
Example 3.2.4. Let K_n be the long virtual knot represented by the Gauss diagram G_n as shown in Fig. 3.2.5. We perform a forbidden move on the rightmost two chords in G_n, and then these two chords are removed. By repeated use of this move n times, the Gauss diagram G_n may be changed to the Gauss diagram without chords.

Moreover, the subdiagram of G_n with non-split two chords is only

. The Gauss diagram G_n has n subdiagrams corresponding to

. Then we have

$v_{2,1}(K_n) = \sum_{\varepsilon_1,\varepsilon_2} \varepsilon_1\varepsilon_2 \Big($  $, I(G_n)\Big)$

$= \Big($  $, n$  $\Big)$

$= n.$

Therefore $u_F(K_n) = n$.

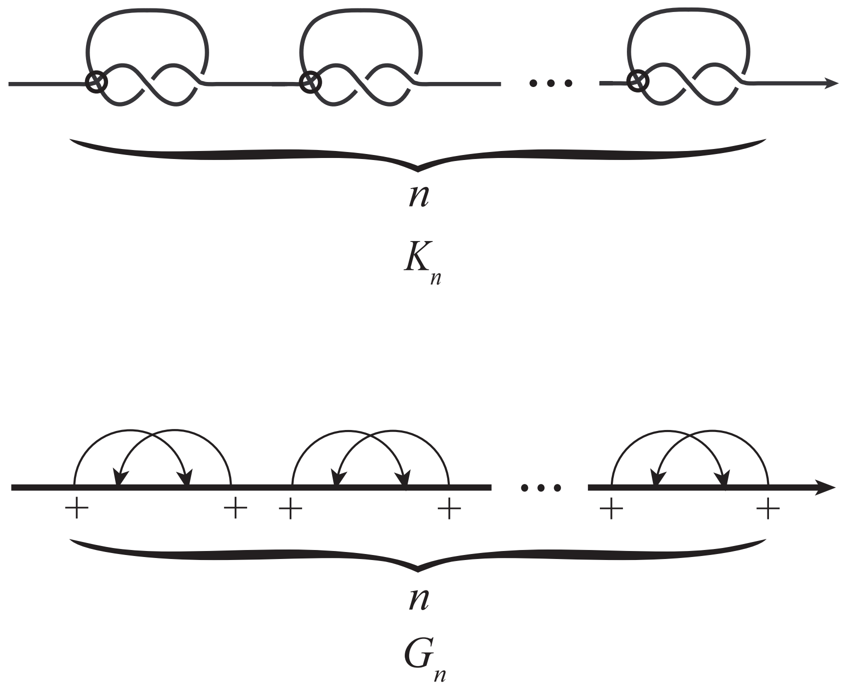


Figure 3.2.5

We prepare some notations for Lemma 3.2.5. Choose distinct points a, b, c, and d on the circle of a Gauss diagram of a virtual knot. When we walk on the circle along the orientation from a to b, we denote the arc that we trace by ab. Define $N^+_{ab,cd}$ as the number of positive chords with tails in ab and heads in cd, and $N^-_{ab,cd}$ as the number of negative chords with tails in ab and heads in cd. Put $N_{ab,cd} = N^+_{ab,cd} - N^-_{ab,cd}$.

Lemma 3.2.5. *The Gauss diagrams $G_1^{\varepsilon\varepsilon'}$ and $G_3^{\varepsilon\varepsilon'}$ are transformed into $G_2^{\varepsilon\varepsilon'}$*

and $G_4^{\varepsilon\varepsilon'}$ *by a single forbidden move respectively, and points* a*,* b*,* c *and* d *are end points of arrows as shown in Fig.* 3.2.6*. Then we have*

$$
\begin{aligned}
&v_{3,1}(G_1^{\varepsilon\varepsilon'}) - v_{3,1}(G_2^{\varepsilon\varepsilon'})\\
&= \begin{cases} -N_{ab,bc} + N_{bc,ab} + N_{bc,cd} - N_{cd,bc} - 1 & (\varepsilon, \varepsilon' = +1),\\ N_{ab,bc} - N_{bc,ab} - N_{bc,cd} + N_{cd,bc} & (\varepsilon \neq \varepsilon'),\\ -N_{ab,bc} + N_{bc,ab} + N_{bc,cd} - N_{cd,bc} + 1 & (\varepsilon, \varepsilon' = -1). \end{cases}\\
&v_{3,1}(G_3^{\varepsilon\varepsilon'}) - v_{3,1}(G_4^{\varepsilon\varepsilon'})\\
&= \begin{cases} N_{ab,bc} - N_{bc,ab} - N_{bc,cd} + N_{cd,bc} - 1 & (\varepsilon, \varepsilon' = +1),\\ -N_{ab,bc} + N_{bc,ab} + N_{bc,cd} - N_{cd,bc} & (\varepsilon \neq \varepsilon'),\\ N_{ab,bc} - N_{bc,ab} - N_{bc,cd} + N_{cd,bc} + 1 & (\varepsilon, \varepsilon' = -1). \end{cases}
\end{aligned}
$$

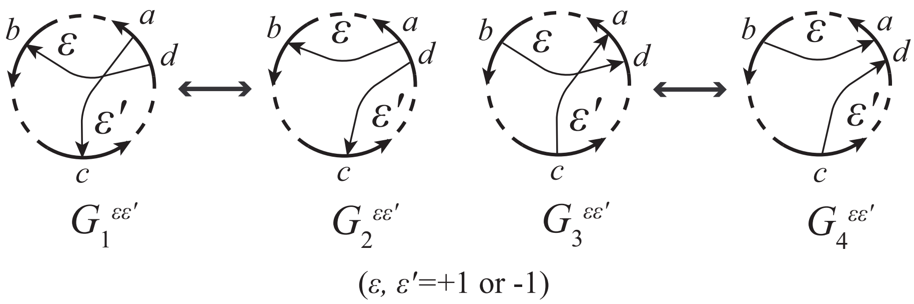
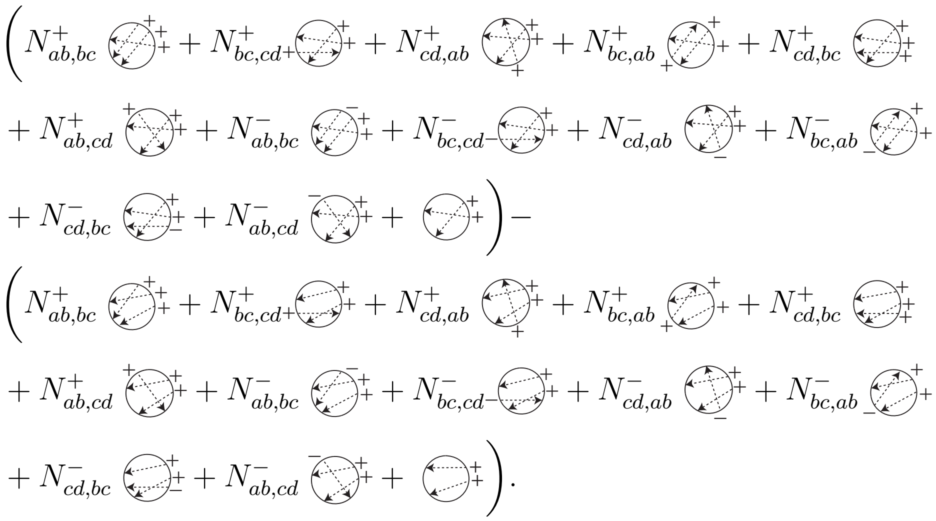


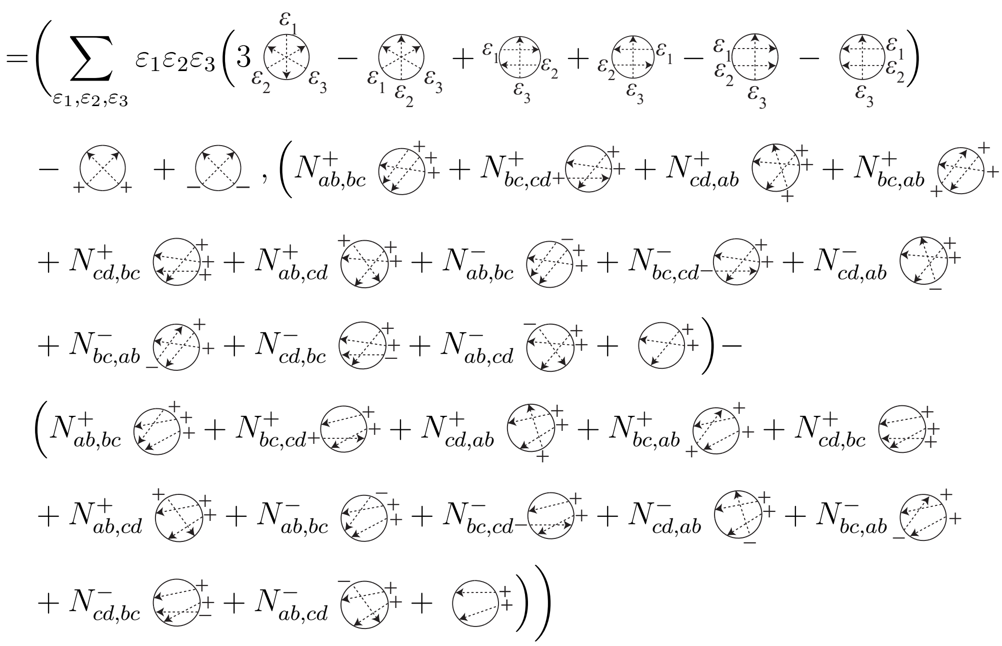
Figure 3.2.6

Proof. We consider the Gauss diagrams G_1^{++} and G_2^{++}. Let ℓ_1 and ℓ_2 be the two chords in the part where a forbidden move is applied. In a similar way as Theorem 3.2.2, we check the terms corresponding to subdiagrams which consist of up to three chords and have both ℓ_1 and ℓ_2 in $I(G_1^{++}) - I(G_2^{++})$. These terms are the following:



Therefore we have

$$v_{3,1}(G_1^{++}) - v_{3,1}(G_2^{++})$$



$$
\begin{aligned}
= &- (N^{+}_{ab,bc} - N^{-}_{ab,bc} - N^{+}_{cd,ab} + N^{-}_{cd,ab}) + (N^{+}_{bc,cd} - N^{-}_{bc,cd}) - (N^{+}_{cd,ab} - N^{-}_{cd,ab} \\
&+ N^{+}_{ab,cd} - N^{-}_{ab,cd}) + (N^{+}_{bc,ab} - N^{-}_{bc,ab}) - (N^{+}_{cd,bc} - N^{-}_{cd,bc} - N^{+}_{ab,cd} + N^{-}_{ab,cd}) \\
&- 1 \\
= &- N_{ab,bc} + N_{bc,ab} + N_{bc,cd} - N_{cd,bc} - 1
\end{aligned}
$$

The other cases are similarly shown.

□

Theorem 3.2.6. *There exists a pair of virtual knots K and K' which satisfies the following for any natural number n:*

(i) $v_{3,1}(K) - v_{3,1}(K') = n$ and

(ii) If Gauss diagrams G and G' are those of K and K' respectively, G and G' can be transformed into each other by a single forbidden move.

Proof. Let K_n be the virtual knot represented by the Gauss diagram G_n as shown in Fig. 3.2.7. The Gauss diagrams G_n and G_{n-1} can be transformed into each other by a single forbidden move. By Lemma 3.2.5, $v_{3,1}(G_n) - v_{3,1}(G_{n-1}) = n$. Therefore, K_n and K_{n-1} satisfy the conditions (i) and (ii).

□

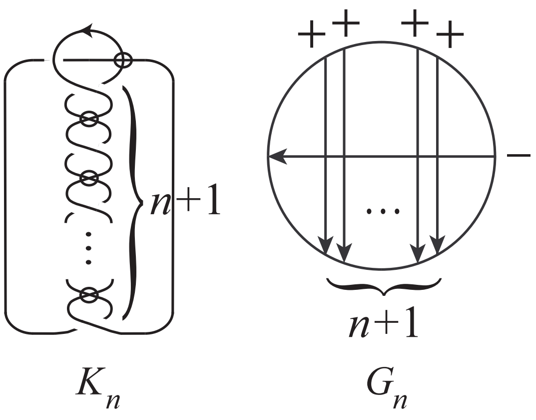


Figure 3.2.7

Theorem 3.2.7. *Let G and G' be Gauss diagrams of long virtual knots K and K' which can be transformed into each other by a single* 3*-variation*(3) *respectively. Then we have*

$$v_{3,1}(K) - v_{3,1}(K') = \pm 1.$$

Proof. Let K and K' be virtual knots represented by the Gauss diagrams G and G' respectively as shown in Fig. 3.2.8 and Fig. 3.2.9. Gauss diagrams G and G' are transformed into each other by forbidden moves via G_f and G'_f as in Fig. 3.2.9. We consider orientations and connecting relations of strings of G and G'. For orientations, it is enough to consider the cases in Fig. 3.2.10, and the connecting relations in the case (I) are six cases in Fig. 3.2.11. We only show the case (I)(i) in Fig. 3.2.12 since the other cases can be treated similarly. If $\varepsilon = +1$, then

$$v_{3,1}(G) = v_{3,1}(G_f) - N_{ab,bc} + N_{bc,ab} + N_{bc,cd} - N_{cd,bc} - 1,$$

$$v_{3,1}(G') = v_{3,1}(G'_f) - (N_{ab,bc} + 1) + N_{bc,ab} + N_{bc,cd} - N_{cd,bc} - 1.$$

Since the Gauss diagrams G_f and G'_f are equivalent, $v_{3,1}(G) - v_{3,1}(G') = 1$. Similarly, if $\varepsilon = -1$, $v_{3,1}(G) - v_{3,1}(G') = -1$

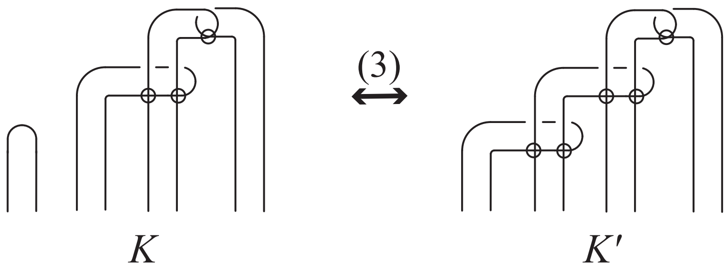


Figure 3.2.8 Virtual knots K and K'

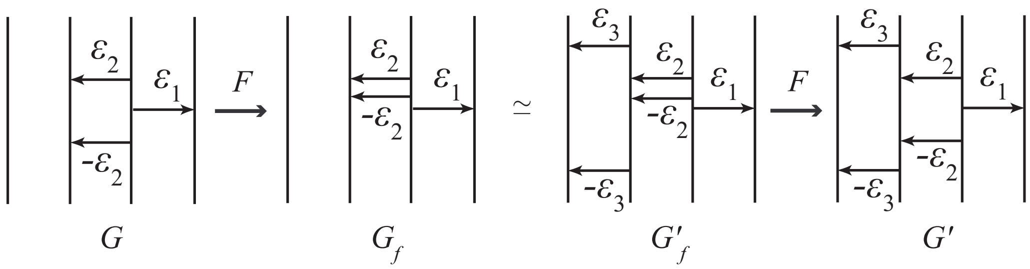


Figure 3.2.9 Gauss diagrams G, G', G_f and G'_f

□

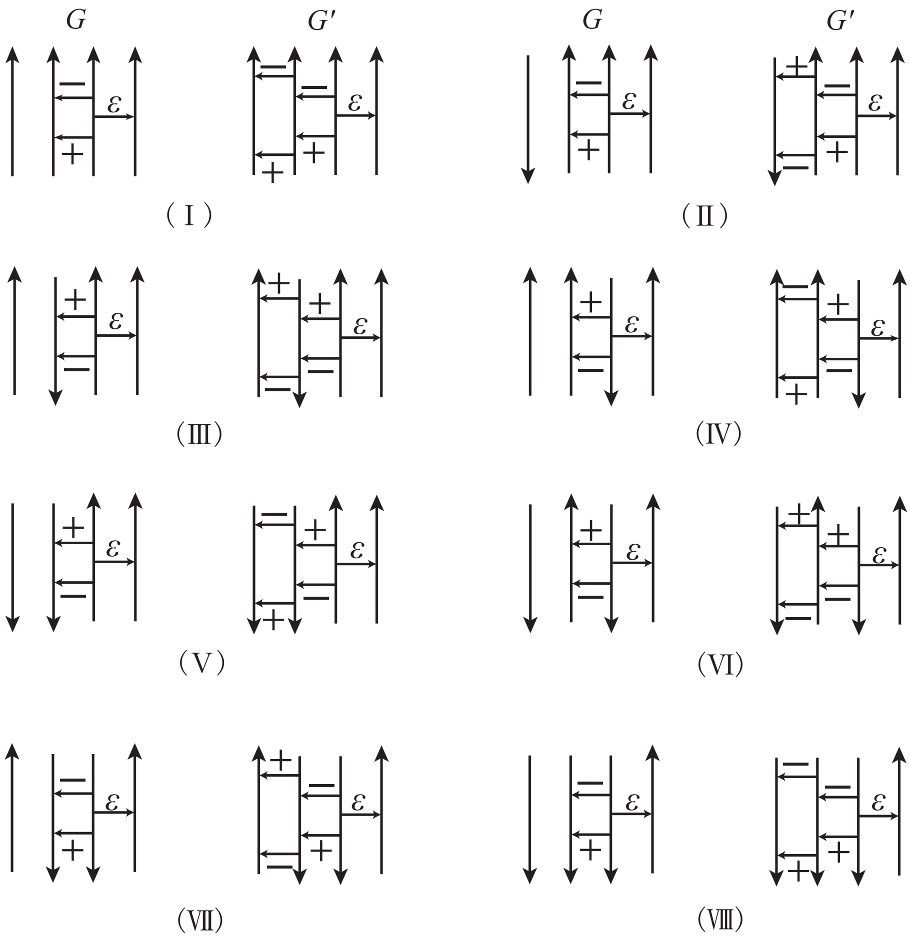


Figure 3.2.10 Oriented Gauss diagrams G and G'

Corollary 3.2.8. *Let K and K' be virtual knots. Then,*

$$d_{(3)}(K,K') \geq \left|v_{3,1}(K) - v_{3,1}(K')\right|.$$

Example 3.2.9. The virtual knots K_n and K'_n in Fig. 3.2.13 are represented by the Gauss diagrams G_n^0 and G_n^n in Fig. 3.2.14 respectively. The Gauss

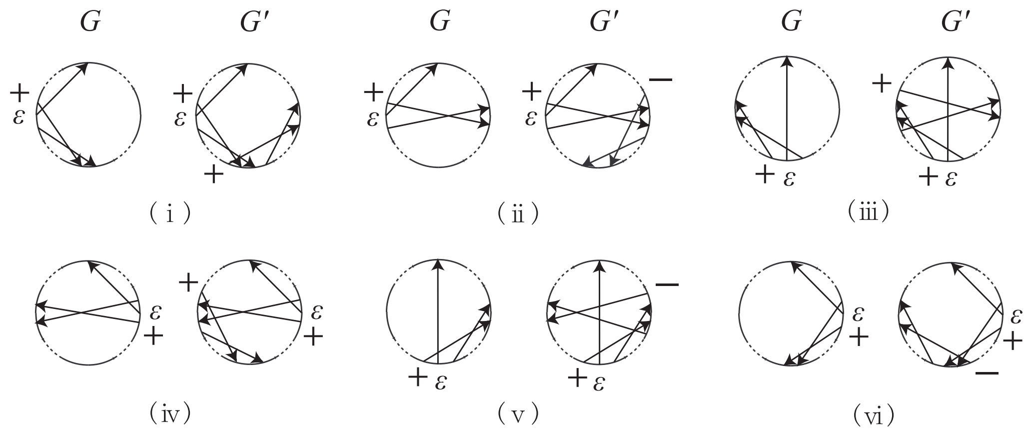


Figure 3.2.11 Connecting relations of strings

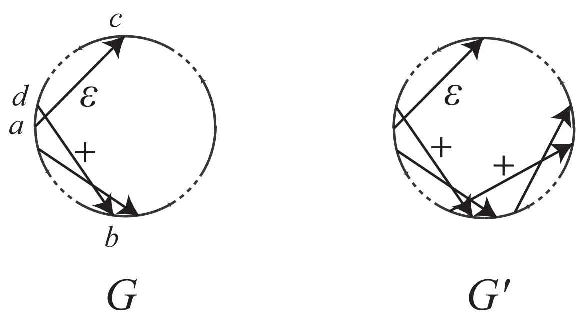


Figure 3.2.12 Labeled G and G'

diagram G_n^n is obtained from G_n^0 by applying 3-variation(3)'s n times, as in Fig. 3.2.14. By Theorem 3.2.7,

$$v_{3,1}(G_n^i) - v_{3,1}(G_n^{i+1}) = 1 \quad (i = 0, 1, \ldots, n-1).$$

Thus, $v_{3,1}(G_n^0) - v_{3,1}(G_n^n) = n$. Form Corollary 3.2.8, we have $d_{(3)}(K_n, K'_n) = n$.

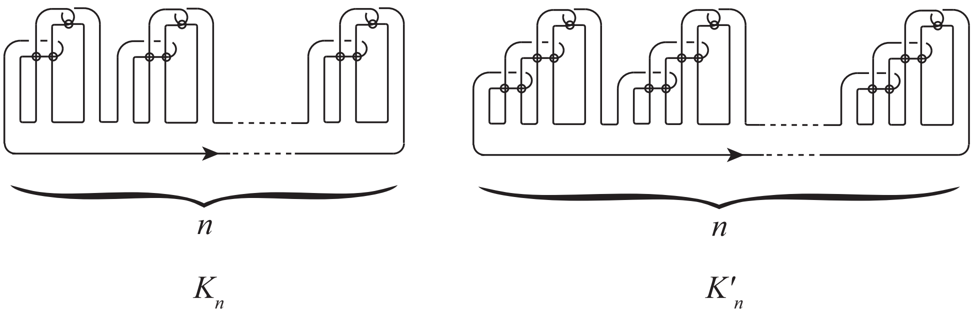


Figure 3.2.13

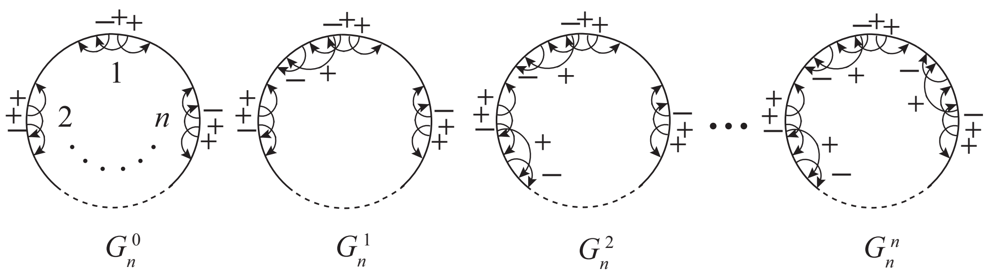


Figure 3.2.14

3.3 An n-variation(n) and Henrich's polynomial invariant

Theorem 3.3.1. *Let D and D' be virtual knot diagrams of virtual knots K and K' which can be transformed into each other by a single n-variation(n) ($n \geq 3$). Then we have*

$$\mathbf{p}_t(K) - \mathbf{p}_t(K') = (t-1)(\pm t^k \pm t^l) \qquad (n = 3) \text{ and}$$

$$\mathbf{p}_t(K) - \mathbf{p}_t(K') = 0 \qquad (n \geq 4)$$

where k and ℓ are some integers.

Proof. We show the case $n = 3$. Virtual knots K and K' have diagrams D and D' in Fig. 3.3.1. Let c be a real crossing of D except c_1, c_2, c_3 and c_4, and c' the crossing of D' corresponding to c. Denote by $\widetilde{d}$ the flat crossing corresponding to a real crossing d.

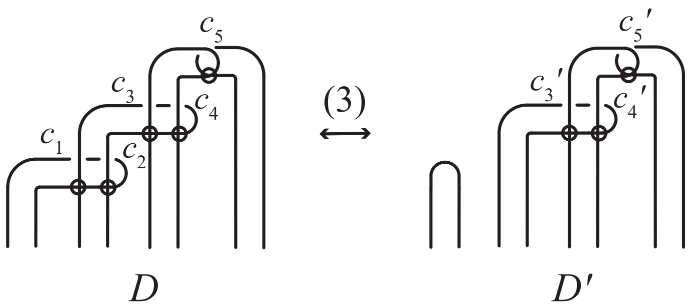


Figure 3.3.1 Virtual knot diagrams D and D'

First, consider the terms of $\mathbf{p}_t(K)$ and $\mathbf{p}_t(K')$ corresponding to c and c' respectively. Comparing $i(c)$ with $i(c')$, $i(c) = i(c')$ or $i(c')+\mathrm{sgn}(\widetilde{c_1})+\mathrm{sgn}(\widetilde{c_2})$. Since $\mathrm{sgn}(\widetilde{c_1}) + \mathrm{sgn}(\widetilde{c_2}) = 0$, $|i(c)| = |i(c')|$. Therefore, $\mathrm{sign}(c)(t^{|i(c)|} - 1) = \mathrm{sign}(c')(t^{|i(c')|} - 1)$. Then, consider the terms of $\mathbf{p}_t(K)$ corresponding to c_1 and c_2. Figure 3.3.2 indicates all cases of orientations for the 3rd and the 4th strings of D, and Fig. 3.3.3 shows D^{c_1} and D^{c_2} in the case (I) in Fig. 3.3.2. We only show the case (I) since the other cases can be treated similarly. Comparing $i(c_1)$ with $i(c_2)$, $i(c_1) + \mathrm{sgn}(\widetilde{c_1}) = i(c_2) + \mathrm{sgn}(\widetilde{c_2}) + \varepsilon\mathrm{sgn}(\widetilde{c_3}) + \varepsilon\mathrm{sgn}(\widetilde{c_4})$ ($\varepsilon = \pm 1$). Here, $\mathrm{sgn}(\widetilde{c_3}) + \mathrm{sgn}(\widetilde{c_4}) = 0$ and $\mathrm{sgn}(\widetilde{c_1}) = \mathrm{sgn}(\widetilde{c_2})$ as shown in Fig. 3.3.3. Thus, $|i(c_1)| = |i(c_2)|$. Since $\mathrm{sign}(c_1) = -\mathrm{sign}(c_2)$, $\mathrm{sign}(c_1)(t^{|i(c_1)|} - 1) + \mathrm{sign}(c_2)(t^{|i(c_2)|} - 1) = 0$.

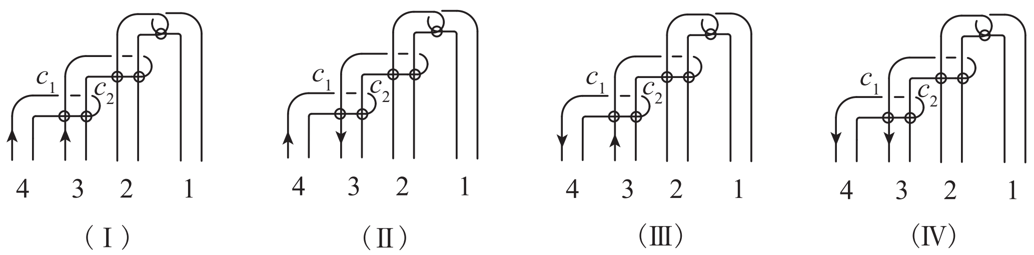


Figure 3.3.2 Orientations of 3rd and 4th strings for D and D'

Finally, consider the terms of $\mathbf{p}_t(K)$ and $\mathbf{p}_t(K')$ corresponding to c_3, c_4, c_3', and c_4'. We may consider the four cases for orientations of the 2nd and the 3rd strings of D and D'. Just as above, we only show the case in Fig. 3.3.4.

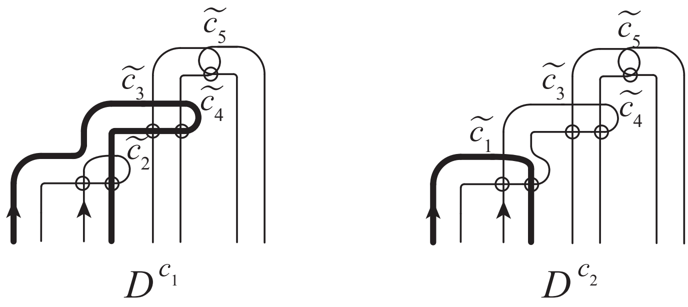


Figure 3.3.3 Flat virtual links D^{c_1} and D^{c_2}

Figure 3.3.5 shows D^{c_3}, $D^{c'_3}$, D^{c_4} and $D^{c'_4}$ in this case. If the string ℓ_1 belongs to the same component as the string ℓ_3 in D^{c_3} and D^{c_4}, the flat crossing $\widetilde{c_5}$ does not contribute $i(c_3)$ and the flat crossing $\widetilde{c_5}$ contributes $i(c_4)$ since the strings ℓ_1 and ℓ_2 belong to different components each other. Similarly, if the string ℓ_1 belongs to the different component from the string ℓ_3, the flat crossing $\widetilde{c_5}$ contributes $i(c_3)$ and the flat crossing $\widetilde{c_5}$ does not contribute $i(c_4)$. Thus, comparing $i(c_3)$ with $i(c_4)$, $i(c_3)+\mathrm{sgn}(\widetilde{c_3}) = i(c_4)+\mathrm{sgn}(\widetilde{c_4})+\varepsilon\mathrm{sgn}(\widetilde{c_5})$. Since $\mathrm{sgn}(\widetilde{c_3}) = \mathrm{sgn}(\widetilde{c_4})$, $|i(c_3)| = |i(c_4) \pm 1|$. Similarly, $|i(c'_3)| = |i(c'_4) \pm 1|$. Since the strings ℓ_1 and ℓ_2 belong to different components in D^{c_3} and $D'^{c'_3}$, the flat crossing $\widetilde{c_1}$ contributes $i(c_3)$ if and only if the flat crossing $\widetilde{c_2}$ does not contribute $i(c_3)$. Comparing $i(c_3)$ with $i(c'_3)$, $i(c_3) = i(c'_3) + \mathrm{sgn}(\widetilde{c_1})$ or $i(c'_3) + \mathrm{sgn}(\widetilde{c_2})$. Thus, $|i(c_3)| = |i(c'_3) \pm 1|$. Therefore we have

$$\left\{\mathrm{sign}(c_3)(t^{|i(c_3)|}-1)+\mathrm{sign}(c_4)(t^{|i(c_4)|}-1)\right\}$$
$$-\left\{\mathrm{sign}(c'_3)(t^{|i(c'_3)|}-1)+\mathrm{sign}(c'_4)(t^{|i(c'_4)|}-1)\right\}$$
$$=\mathrm{sign}(c_3)t^{|i(c_3)|}(1-t^{\pm 1})-\mathrm{sign}(c'_3)t^{|i(c'_3)|}(1-t^{\pm 1})$$
$$=-t^{|i(c_3)|}(1-t^{\pm 1})+t^{|i(c_3)|\pm 1}(1-t^{\pm 1})$$
$$=\begin{cases}(t-1)(t^{|i(c_3)|}-t^{|i(c_3)|\pm 1}),\\(t-1)(-t^{|i(c_3)|-1}-t^{|i(c_3)|\pm 1}),\\(t-1)(t^{|i(c_3)|}+t^{|i(c_3)|}),\\(t-1)(t^{|i(c_3)|}+t^{|i(c_3)|-2}),\\(t-1)(-t^{|i(c_3)|-1}+t^{|i(c_3)|}),\\(t-1)(-t^{|i(c_3)|-1}+t^{|i(c_3)|-2}).\end{cases}$$

Thus, we obtain $\mathbf{p}_t(K)-\mathbf{p}_t(K')=(t-1)(\pm t^k \pm t^l)$.

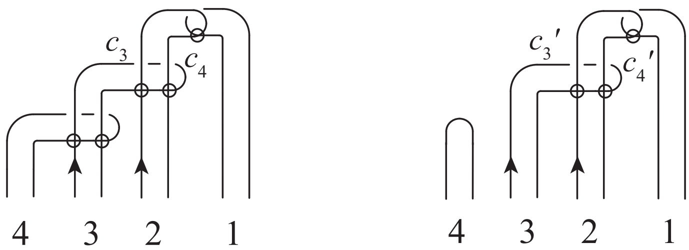


Figure 3.3.4

We show the case $n \geq 4$ as shown in Fig. 3.3.6. We may prove just by the same way as the case $n=3$ except $i(c_3)$, $i(c_4)$, $i(c'_3)$ and $i(c'_4)$. Comparing $i(c_3)$ with $i(c_4)$, $i(c_3)+\mathrm{sgn}(\widetilde{c_3})=i(c_4)+\mathrm{sgn}(\widetilde{c_4})+\varepsilon\mathrm{sgn}(\widetilde{c_5})+\varepsilon\mathrm{sgn}(\widetilde{c_6})$. Since $\mathrm{sgn}(\widetilde{c_5})+\mathrm{sgn}(\widetilde{c_6})=0$, $|i(c_3)|=|i(c_4)|$. Since $\mathrm{sign}(c_3)=-\mathrm{sign}(c_4)$, $\mathrm{sign}(c_3)(t^{|i(c_3)|}-1)+\mathrm{sign}(c_4)(t^{|i(c_4)|}-1)=0$. Similarly, $\mathrm{sign}(c'_3)(t^{|i(c'_3)|}-1)+$

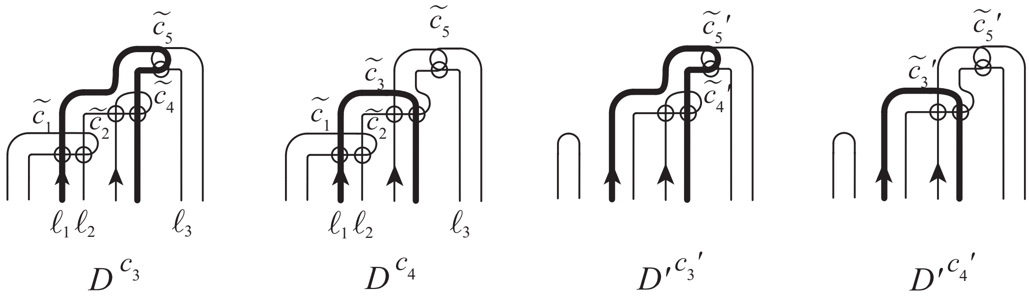


Figure 3.3.5 Flat virtual links D^{c_3}, D^{c_4}, $D'^{c'_3}$ and $D'^{c'_4}$

$\mathrm{sign}(c'_4)(t^{|i(c'_4)|} - 1) = 0$. Therefore, $\mathbf{p}_t(K) - \mathbf{p}_t(K') = 0$.

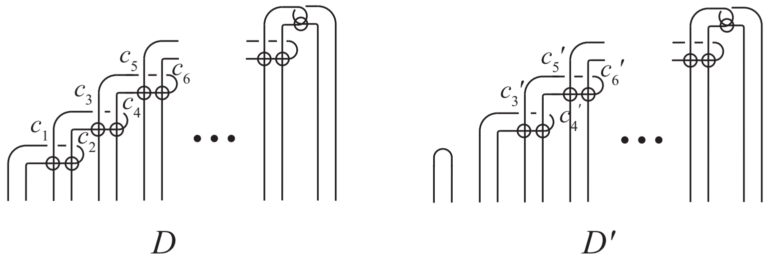


Figure 3.3.6 Virtual knot diagram D and D'

□

Corollary 3.3.2. *Let K and K' be virtual knots, and $\mathbf{p}_t(K) - \mathbf{p}_t(K') = (t-1)\sum_{j\geq 0} a_j t^j$. Then,*

$$d_{(3)}(K, K') \geq \frac{\sum_{j\geq 0} |a_j|}{2}.$$

Example 3.3.3. For $n \in \mathbb{N}$, let K_n be a virtual knot represented by the diagram D_n as shown in Fig. 3.3.7. If n is even, flat virtual links $D_n^{c_i}$ ($i = 1, 2, \ldots, n$) are shadows of the virtual Hopf link, and $D_n^{c_{n+1}}$ is a shadow of the trivial link. Since $|i(c_i)| = 1$ and $|i(c_{n+1})| = 0$, $\mathbf{p}_t(K_n) = n(t-1)$. On the other hand, if n is odd, both of $D_n^{c_i}$ and $D_n^{c_{n+1}}$ are shadows of the virtual Hopf link. Since $|i(c_i)| = |i(c_{n+1})| = 1$, $\mathbf{p}_t(K_n) = (n+1)(t-1)$. Since $\sum_{j \geq 0} |a_j|$ is non-zero, K_n cannot be transformed into the trivial knot by n-variation(n)'s $(n \geq 4)$. Therefore an n-variation(n) $(n \geq 4)$ is not an unknotting operation.

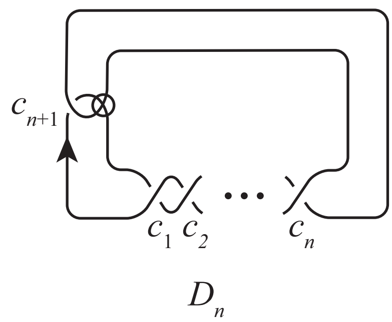
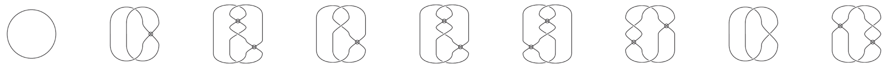


Figure 3.3.7

Virtual knots with up to 3 crossings



0.1 2.1 3.1 3.2 3.3 3.4 3.5 3.6 3.7

Virtual knots 4 crossings

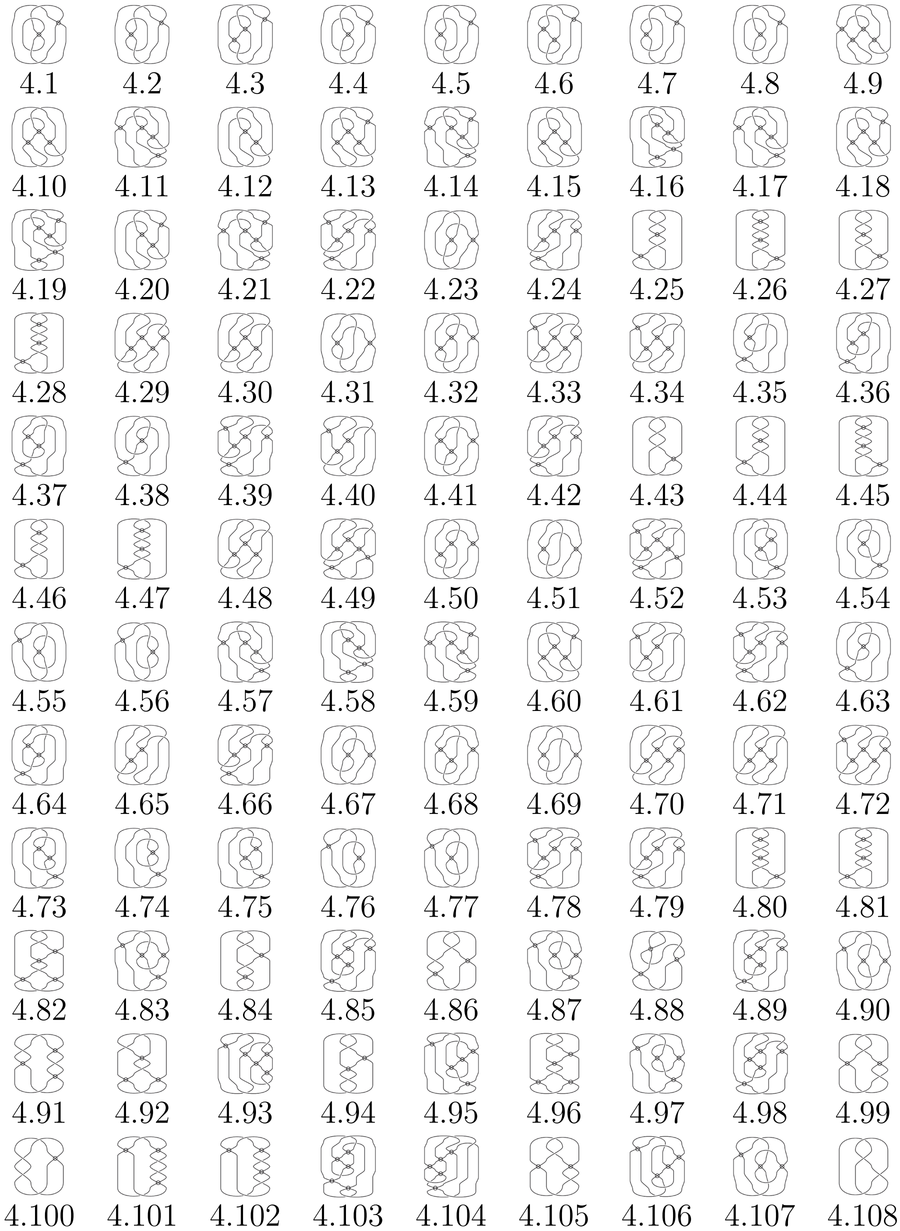


Table 3.3.1 Virtual knot table [4]

Bibliography

[1] E. Byberi and V. V. Chernov, Virtual bridge number one knots, *Commun. Contemp. Math.* **10** (Suppl. 1) (2008) 1013—1021.

[2] S. Chmutov, S. Duzhin and J. Mostovoy, *Introduction to Vassiliev Knot Invariants* (Cambridge University Press, 2012).

[3] M. Goussarov, M. Polyak and O. Viro, Finite type invariants of classical and virtual knots, *Topology* **39** (5) (2000) 1045—1068.

[4] J. Green, A table of virtual knots, http://www.math.toronto.edu/drorbn/Students/GreenJ/.

[5] A. Henrich, A sequence of degree one Vassiliev invariants for virtual knots, *J. Knot Theory Ramications* **19** (4) (2010) 461—487.

[6] M. Hirasawa, N. Kamada and S. Kamada, Bridge presentations of virtual knots, *J. Knot Theory Ramications* **20** (6) (2011) 881—893.

[7] T. Kanenobu, Forbidden moves unknot a virtual knot, *J. Knot Theory Ramications* **10** (1) (2001) 89—96.

[8] L. H. Kauffman, Virtual knot theory, *European J. Combin.* **20** (7) (1999) 663—691.

[9] K. Murasugi, Knot Theory and Its Applications, *Birkhuser* (1996) (translated from the 1993 Japanese original by B. Kurpita).

[10] S. Nelson, Unknotting virtual knots with Gauss diagram forbidden moves, *J. Knot Theory Ramications* **10** (6) (2001) 931—935.

[11] Y. Ohyama and T. Tsukamoto, On Habiro ' s C_n-moves and Vassiliev invariants of order n, *J. Knot Theory Ramications* **8** (1) (1999) 15—23.

[12] T. Tsukamoto, Clasp-pass moves and Vassiliev invariants of type three for knots, *Proc. Amer. Math. Soc.* **128** (6) (2000) 1859—1867.

[13] V. A. Vassiliev, Cohomology of knot spaces, in *Theory of Singularities and Its Applications*, Advances in Soviet Mathematics, Vol. 1 (American Mathematical Society, Providence, RI, 1990), pp. 23—69.

List of papers by Migiwa Sakurai

[1] An estimate of the unknotting numbers for virtual knots by forbidden moves, *J. Knot Theory Ramifications* **22** (3) (2013).

[2] 2 and 3-variations and finite type invariants of degree 2 and 3, *J. Knot Theory Ramifications* **22** (8) (2013).