A GENERALIZATION OF MINTY'S THEOREM

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1. Introduction

In the evolution equation theory in Hilbert spaces, the following Minty's theorem [3] for monotone operators is most important :

(I) *A monotone operator A in a Hilbert space H is maximal if and only if $R(I + \lambda A) = H$ for some $\lambda > 0$.*

(II) *If a singlevalued monotone operator A in a Hilbert space H is defined on H and continuous, then A is maximal.*

This theorem is effectively used to generate nonlinear semigroups or to show the existence of solutions to nonlinear elliptic equations of some type. This theorem has been generalized to Banach spaces (for instance Barbu [1]). However, Minty's theorem does not seem to have been generalized to a wider class than that of monotone operators till Shimizu [4].

In 1992 Shimizu introduced a class of operators in a Hilbert space denoted by $\mathcal{M}(\alpha)$, a generalization of monotone operators, and showed that Minty's theorem holds in this class.

In this paper we intend to introduce a class of operators in a Hilbert space denoted by $\mathcal{M}(\{\alpha_\lambda\})$, a generalization of $\mathcal{M}(\alpha)$, and prove that Minty's theorem still holds in this class. Furthermore we discuss perturbation theorems on operators in the class $\mathcal{M}(\{\alpha_\lambda\})$.

2. Class $\mathcal{M}(\{\alpha_\lambda\})$

Let H be a real or complex Hilbert space with norm $\|\cdot\|$ and inner-product $\langle\cdot\,,\cdot\rangle$.

We consider possibly multivalued nonlinear operators in H. For a such operator A, we denote its domain by $D(A)$, its range by $R(A)$ and its inverse operator by A^{-1}. The domain $D(A+B)$ of the sum $A+B$ is understood as $D(A) \cap D(B)$.

The monotone operators are defined as follows :

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Definition 1 An operator A in H is said to be *monotone,* if A satisfies the following condition :

$$Re\langle x_1' - x_2',\, x_1 - x_2\rangle \geqq 0$$

for any $x_1, x_2 \in D(A)$, $x_1' \in Ax_1$, $x_2' \in Ax_2$.

The class $\mathcal{M}(\alpha)$, given by Shimizu [4], is defined as follows :

Definition 2 (Shimizu) Let $\alpha \in [0,1)$. An operaor A in H is said to be *of class* $\mathcal{M}(\alpha)$, if A satisfies the following condition :

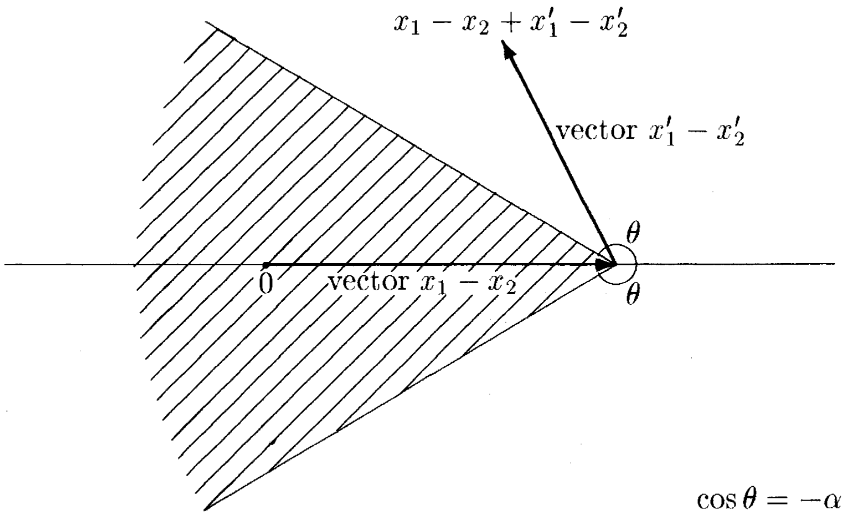
$$Re\langle x_1' - x_2',\, x_1 - x_2\rangle \geqq -\alpha \, \|x_1' - x_2'\| \, \|x_1 - x_2\|$$

for any $x_1, x_2 \in D(A)$, $x_1' \in Ax_1$, $x_2' \in Ax_2$.

Evidently, we have $\mathcal{M}(\alpha) \subset \mathcal{M}(\beta)$ for $0 \leqq \alpha < \beta < 1$, and A is monotone if and only if $A \in \mathcal{M}(0)$.

We shall give a geometrical interpretation of an operator of class $\mathcal{M}(\alpha)$. In the case of real Hilbert space H, $A \in \mathcal{M}(\alpha)$ means as follows (Figure 1). When any $x_1, x_2 \in D(A)$ are fixed, for any $x_1' \in Ax_1$, $x_2' \in Ax_2$, the vector $x_1' - x_2'$ makes an angle with the vector $x_1 - x_2$ whose cosine is greater than $-\alpha$. In other words, the point $x_1 - x_2 + x_1' - x_2'$ does not belong to the shadow part in Figure 1.

Figure 1



On the operators of class $\mathcal{M}(\alpha)$, Shimizu [4] proved the following proposition :

An operator A is of class $\mathcal{M}(\alpha)$ if and only if

$$\|x_1 - x_2\| \leqq (1-\alpha^2)^{-\frac{1}{2}} \|x_1 - x_2 + \lambda(x_1' - x_2')\|$$

for any $x_1, x_2 \in D(A)$, $x_1' \in Ax_1$, $x_2' \in Ax_2$ and any $\lambda > 0$.

Suggested by the fact mentioned above, we introduce the following class of operators as a generalization of class $\mathcal{M}(\alpha)$:

Definition 3 Let $\{\alpha_\lambda\}_{\lambda>0}$ be a family of real numbers $\alpha_\lambda \geqq 1$ for all $\lambda > 0$. An operator A in H is said to be *of class* $\mathcal{M}(\{\alpha_\lambda\})$, if A satisfies the following condition :

$$\|x_1 - x_2\| \leqq \alpha_\lambda \|x_1 - x_2 + \lambda(x_1' - x_2')\|$$

for any $x_1, x_2 \in D(A)$, $x_1' \in Ax_1$, $x_2' \in Ax_2$ and any $\lambda > 0$.

A geometrical meaning of an operator belonging to the class $\mathcal{M}(\{\alpha_\lambda\})$ is given by the next lemma :

Lemma 1 *An operator A is of class $\mathcal{M}(\{\alpha_\lambda\})$ if and only if A satisfies the following condition :*

For any $x_1, x_2 \in D(A)$ and any $\lambda > 0$, $U_\lambda(x_1, x_2)$ denotes an open ball with the center $\left(1 - \dfrac{1}{\lambda}\right)(x_1 - x_2)$ and the radius $\dfrac{\|x_1 - x_2\|}{\lambda\,\alpha_\lambda}$, and $U(x_1, x_2) = \bigcup_{\lambda>0} U_\lambda(x_1, x_2)$. Then,

$$x_1 - x_2 + x_1' - x_2' \notin U(x_1, x_2)$$

for any $x_1' \in Ax_1$, $x_2' \in Ax_2$.

Proof. If $A \in \mathcal{M}(\{\alpha_\lambda\})$, then for any $x_1, x_2 \in D(A)$, $x_1' \in Ax_1$, $x_2' \in Ax_2$ and $\lambda > 0$

$$\begin{aligned}&\left\|(x_1 - x_2 + x_1' - x_2') - \left(1 - \frac{1}{\lambda}\right)(x_1 - x_2)\right\| \\ &\quad = \frac{1}{\lambda}\|x_1 - x_2 + \lambda(x_1' - x_2')\| \geq \frac{\|x_1 - x_2\|}{\lambda\,\alpha_\lambda},\end{aligned}$$

which implies

$$x_1 - x_2 + x_1' - x_2' \notin U_\lambda(x_1, x_2).$$

Hence

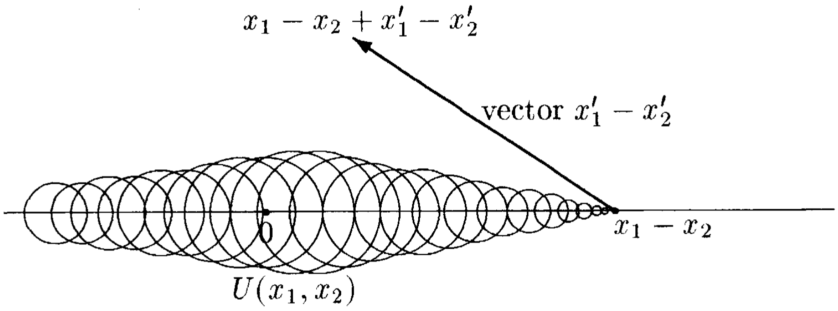
$$x_1 - x_2 + x_1' - x_2' \notin U(x_1, x_2).$$

Similarly, the inverse implication holds. □

In the case of real Hilbert space H, we illustrate Lemma 1 by Figure 2. Let $A \in \mathcal{M}(\{\alpha_\lambda\})$ and $x_1, x_2 \in D(A)$. When λ runs over $(0,\infty)$, the center $(1-\frac{1}{\lambda})(x_1-x_2)$ of the open ball $U_\lambda(x_1,x_2)$ moves on the half line $(-\infty(x_1-x_2),\ x_1-x_2)$. The center tends to $-\infty(x_1-x_2)$ as $\lambda \to 0$, and converges to x_1-x_2 as $\lambda \to \infty$.

We now consider $\tilde{A} \in \mathcal{M}(\alpha)$ for any $\alpha \in [0,1)$. It should be noted that we can choose $\{\alpha_\lambda\}_{\lambda>0}$ $(\alpha_\lambda \geqq 1)$ such that $U(x_1,x_2)$ in Figure 2 is contained in the shadow part in Figure 1 for $\tilde{A}$, however near to π, $\theta = \arccos(-\alpha)$ may be.

Figure 2



3. Main Theorems

Lemma 2 *For $A \in \mathcal{M}(\{\alpha_\lambda\})$, $(I+\lambda A)^{-1}$ is a singlevalued operator for any $\lambda > 0$.*

Proof. Fix $\lambda > 0$ arbitrarily. If $x_1, x_2 \in D(A)$, $x_1 \neq x_2$, then

$$(I+\lambda A)x_1 \cap (I+\lambda A)x_2 = \emptyset\,.$$

Indeed, if the conclusion is false, there exist $x_1' \in Ax_1$ and $x_2' \in Ax_2$ such that $x_1 + \lambda x_1' = x_2 + \lambda x_2'$. Hence

$$0 < \|x_1 - x_2\| \leqq \alpha_\lambda \|x_1 - x_2 + \lambda(x_1' - x_2')\| = 0\,,$$

which is absurd. Therefore, for any fixed $x \in D(A)$ we have

$$(I+\lambda A)^{-1}y = x \quad \text{for any } y \in (I+\lambda A)x\,.$$ □

That an operator A is of class $\mathcal{M}(\{\alpha_\lambda\})$, means that for any $\lambda > 0$

$$\|(I + \lambda A)^{-1} y_1 - (I + \lambda A)^{-1} y_2\| \leqq \alpha_\lambda \|y_1 - y_2\|$$

for any $y_1, y_2 \in D((I + \lambda A)^{-1})$.

Definition 4 Let $\beta > 0$. A singlevalued operator T in H is said to be *of class $\mathcal{L}(\beta)$*, if T is a *Lipschitz continuous operator* with the *Lipschitz constant β*, that is,

$$\|Ty_1 - Ty_2\| \leqq \beta \|y_1 - y_2\| \qquad \text{for any } y_1, y_2 \in D(T).$$

Consequently, an operator A is of class $\mathcal{M}(\{\alpha_\lambda\})$, if and only if $(I + \lambda A)^{-1}$ is of class $\mathcal{L}(\alpha_\lambda)$ for any $\lambda > 0$.

For two operators A and B in H, we denote $A \subset B$ if B is an extension of A, that is, $D(A) \subset D(B)$ and $Ax \subset Bx$ for all $x \in D(A)$.

An operator $A \in \mathcal{M}(\{\alpha_\lambda\})$ is said to be *maximal* in the class $\mathcal{M}(\{\alpha_\lambda\})$, if $B \in \mathcal{M}(\{\alpha_\lambda\})$ and $B \supset A$ imply $B = A$. Maximality in the class $\mathcal{L}(\beta)$ is defined similarly.

As is easily seen, we have the following two lemmas :

Lemma 3 *For $A, B \in \mathcal{M}(\{\alpha_\lambda\})$, the following three conditions are equivalent :*
(1) $A \subset B$,
(2) $(I + \lambda A)^{-1} \subset (I + \lambda B)^{-1}$ *for any* $\lambda > 0$,
(3) $(I + \lambda A)^{-1} \subset (I + \lambda B)^{-1}$ *for some* $\lambda > 0$.

Lemma 4 *For any $A \in \mathcal{M}(\{\alpha_\lambda\})$, the following three conditions are equivalent :*
(1) *A is maximal in $\mathcal{M}(\{\alpha_\lambda\})$,*
(2) *$(I + \lambda A)^{-1}$ is maximal in $\mathcal{L}(\alpha_\lambda)$ for any $\lambda > 0$,*
(3) *$(I + \lambda A)^{-1}$ is maximal in $\mathcal{L}(\alpha_\lambda)$ for some $\lambda > 0$.*

The following proposition is well known :

Proposition 1 *Suppose T is a singlevalued operator in H satisfying*

$$\|Ty_1 - Ty_2\| \leqq \beta \|y_1 - y_2\| \qquad \textit{for any } y_1, y_2 \in D(T).$$

Then there exists at least one singlevalued operator $\tilde{T}$ defined on H such that $\tilde{T} \supset T$ and

$$\|\tilde{T}y_1 - \tilde{T}y_2\| \leqq \beta \|y_1 - y_2\| \qquad \textit{for any } y_1, y_2 \in H.$$

In other words, any operator $T \in \mathcal{L}(\beta)$ has a maximal extension $\tilde{T}$ in $\mathcal{L}(\beta)$ with $D(\tilde{T}) = H$.

This proposition can be proved by making use of *Zorn's lemma*. For the proof we refer to Kōmura and Konishi [2, p.15].

Now we state our main theorems. The first one is a generalization of Minty's theorem (I) :

Theorem 1 *For an operator $A \in \mathcal{M}(\{\alpha_\lambda\})$, the following three conditions are equivalent :*
(1) *A is maximal in the class $\mathcal{M}(\{\alpha_\lambda\})$,*
(2) *$R(I+\lambda A) = H$ for any $\lambda > 0$,*
(3) *$R(I+\lambda A) = H$ for some $\lambda > 0$.*

Proof. (1)$\Rightarrow$(2) Assume A is maximal in $\mathcal{M}(\{\alpha_\lambda\})$. Then, by Lemma 4, $(I+\lambda A)^{-1}$ is maximal in $\mathcal{L}(\alpha_\lambda)$ for any $\lambda > 0$. If (2) does not hold, then for some $\lambda_0 > 0$, $D((I+\lambda_0 A)^{-1}) = R(I+\lambda_0 A) \subsetneq H$. By Proposition 1, $(I+\lambda_0 A)^{-1}$ has a maximal extension in $\mathcal{L}(\alpha_{\lambda_0})$ defined on H, which contradicts the maximality of $(I+\lambda_0 A)^{-1}$.
(2)$\Rightarrow$(3) is trivial.
(3)$\Rightarrow$(1) Suppose $R(I+\lambda_0 A) = H$ for some $\lambda_0 > 0$. Then $(I+\lambda_0 A)^{-1}$ is maximal in $\mathcal{L}(\alpha_{\lambda_0})$. Thus A is maximal in $\mathcal{M}(\{\alpha_\lambda\})$ by Lemma 4. □

The second theorem is a generalization of Minty's theorem (II) :

Theorem 2 *Let $\{\alpha_\lambda\}_{\lambda>0}$ ($\alpha_\lambda \geqq 1$ for all $\lambda > 0$) satisfy the condition that $\liminf_{\lambda\to 0} \alpha_\lambda < \infty$. Let A be a singlevalued operator in the class $\mathcal{M}(\{\alpha_\lambda\})$. If $D(A) = H$ and A is hemicontinuous (that is, continuous along any line segment), then A is maximal in the class $\mathcal{M}(\{\alpha_\lambda\})$.*

Proof. Assume that A is not maximal in $\mathcal{M}(\{\alpha_\lambda\})$. Then A has a maximal extension $\tilde{A}$ in $\mathcal{M}(\{\alpha_\lambda\})$ with $D(\tilde{A}) = H$, since $D(A) = H$. Hence there exist $x \in H$ and $\tilde{y} \in \tilde{A}x$ such that $Ax \neq \tilde{y}$. Put $x_\lambda = x + \lambda(\tilde{y} - Ax)$ for $\lambda > 0$. Since $x_\lambda \to x$ as $\lambda \to 0$ and A is hemicontinuous, $Ax_\lambda \to Ax$ as $\lambda \to 0$. Noting $A \subset \tilde{A}$, $\tilde{A} \in \mathcal{M}(\{\alpha_\lambda\})$, we have

$$\|x_\lambda - x\| \leqq \alpha_\lambda \|x_\lambda - x + \lambda(Ax_\lambda - \tilde{y})\| \qquad \text{for any } \lambda > 0.$$

Substituting $x_\lambda - x = \lambda(\tilde{y} - Ax)$, we get

$$\|\tilde{y} - Ax\| \leqq \alpha_\lambda \|Ax_\lambda - Ax\| \qquad \text{for any } \lambda > 0.$$

Since $\lim_{\lambda\to 0} \|Ax_\lambda - Ax\| = 0$ and $\liminf_{\lambda\to 0} \alpha_\lambda < \infty$, letting $\lambda \to 0$, we get $\|\tilde{y} - Ax\| \leqq 0$, which is a contradiction. □

4. Perturbation Theorems

Finally, we show the following perturbation theorems for operators in the class $\mathcal{M}(\{\alpha_\lambda\})$:

Theorem 3 *Suppose A is a maximal operator in the class $\mathcal{M}(\{\alpha_\lambda\})$ and B is an operator in the class $\mathcal{L}(\beta)$ with $D(B) = H$ such that $\beta < 1/(\lambda_0 \alpha_{\lambda_0})$ for some $\lambda_0 > 0$. If $A + B \in \mathcal{M}(\{\gamma_\lambda\})$ for some $\{\gamma_\lambda\}_{\lambda>0}$ such that $\gamma_\lambda \geqq 1$ for $\lambda > 0$, then $A + B$ is maximal in the class $\mathcal{M}(\{\gamma_\lambda\})$.*

Proof. The maximality of $A + B$ in the class $\mathcal{M}(\{\gamma_\lambda\})$ is equivalent to $R(I + \lambda(A + B)) = H$ for some $\lambda > 0$, by Theorem 1. Hence it suffices to show that the equation $u + \lambda_0(A + B)u \ni f$ has a solution $u \in D(A)$ for any $f \in H$. Since A is maximal, a singlevalued operator T with $D(T) = H$ can be defined by

$$Tg = (I + \lambda_0 A)^{-1}(f - \lambda_0 Bg) \quad \text{for } g \in H.$$

T is a strict contraction operator. Indeed, for any $g_1, g_2 \in H$

$$\begin{aligned}
&\|Tg_1 - Tg_2\| \\
&\quad = \|(I + \lambda_0 A)^{-1}(f - \lambda_0 Bg_1) - (I + \lambda_0 A)^{-1}(f - \lambda_0 Bg_2)\| \\
&\quad \leqq \alpha_{\lambda_0} \|(f - \lambda_0 Bg_1) - (f - \lambda_0 Bg_2)\| \\
&\quad = \alpha_{\lambda_0} \lambda_0 \|Bg_1 - Bg_2\| \leqq \alpha_{\lambda_0} \lambda_0 \beta \, \|g_1 - g_2\|
\end{aligned}$$

and $\alpha_{\lambda_0} \lambda_0 \beta < 1$. Therefore T has a unique fixed point $u \in H$, that is,

$$(I + \lambda_0 A)^{-1}(f - \lambda_0 Bu) = u.$$

Hence $u \in D(A)$ and $f - \lambda_0 Bu \in (I + \lambda_0 A)u$, i.e. $f \in u + \lambda_0(A + B)u$. □

Theorem 4 *Suppose A is an operator in the class $\mathcal{M}(\{\alpha_\lambda\})$ and B is a monotone operator such that $D(A) \cap D(B) \neq \emptyset$. If A and B satisfy the following condition :*

$$\|x_1' - x_2'\| \leqq \|x_1' - x_2' + x_1'' - x_2''\|$$

for any $x_1, x_2 \in D(A) \cap D(B)$, $x_1' \in Ax_1$, $x_2' \in Ax_2$, $x_1'' \in Bx_1$, $x_2'' \in Bx_2$, then $A + B$ is of class $\mathcal{M}(\{\alpha_\lambda\})$.

Proof. For any $x_1, x_2 \in D(A + B) = D(A) \cap D(B)$, $x_1' \in Ax_1$, $x_2' \in Ax_2$, $x_1'' \in Bx_1$, $x_2'' \in Bx_2$ and $\lambda > 0$, by the assumption

$$\begin{aligned}
&\|x_1 - x_2 + \lambda\{(x_1' + x_1'') - (x_2' + x_2'')\}\|^2 \\
&\quad = \|x_1 - x_2\|^2 + 2\lambda \operatorname{Re}\langle (x_1' + x_1'') - (x_2' + x_2''),\ x_1 - x_2\rangle \\
&\qquad + \lambda^2 \|(x_1' + x_1'') - (x_2' + x_2'')\|^2 \\
&\quad \geqq \|x_1 - x_2\|^2 + 2\lambda \operatorname{Re}\langle x_1' - x_2',\ x_1 - x_2\rangle + \lambda^2 \|x_1' - x_2'\|^2 \\
&\quad = \|x_1 - x_2 + \lambda(x_1' - x_2')\|^2 \geqq \|x_1 - x_2\|^2/{\alpha_\lambda}^2,
\end{aligned}$$

hence, $A + B \in \mathcal{M}(\{\alpha_\lambda\})$. □

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